THE
MATHEMATICAL GAZETTE

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RAWDON LEVETT.

SEVERAL contributions to the correspondence columns in *Nature* for 1870 deal with Euclid as a textbook for school geometry. In the issue for 26th May, 1870, the following letter appears :

" There are many engaged in the work of education in this country, besides those who have come prominently forward in the matter, who feel strongly that Geometry as now taught falls far short of being that powerful means of education in the highest sense which it might easily be made. They find themselves, in the majority of cases, compelled to use in their classes a text-book which should long ago have become obsolete.

" We have lately had instances in abundance of the power of combined action. If the leaders of the agitation for the reform of our geometrical teaching would organise an Anti-Euclid Association, I feel sure they would meet with considerable and daily-increasing support.

" We of the rank and file do not feel strong enough to act alone, and yet think we might do something to help forward the good cause by cooperating with others.

" The immediate objects of such an association should be in my opinion (1) To collect and distribute information connected with the subject ; (2) To induce examining bodies to frame their questions in geometry without reference to any particular textbook."

The signatory of this letter, Rawdon Levett, of King Edward's School, Birmingham, may well be termed the Founder of the Mathematical Association ; he served it as a Secretary from 1870 to 1883, and on his resignation the President and the Association expressed their deep sense of the debt owed to him, for his services in setting on foot and nursing into strength the Association foreshadowed by the letter just quoted.

Considerations of space will not permit us to give here any further details of the life and work of a great leader. We may refer readers to the obituary notice in Vol. XI, by C. H. P. Mayo and A. W. Siddons, and to Mr. Siddons' Presidential Address, " Progress ", in Vol. XX. For the plate, we are also indebted to Mr. Siddons.

WAVE AND CLASSICAL MECHANICS.

By J. A. Teegan.

1. The theory of relativity leads to a relation between matter and radiant energy such that a mass m is equivalent to a quantity of energy E given by

$$E = mc^2, \qquad\qquad \text{(i)}$$

in which c is the velocity of light. According to the quantum theory a photon of wavelength λ has energy E given by

$$E = hc/\lambda, \qquad\qquad \text{(ii)}$$

h being Planck's constant. Combining (i) and (ii) we get $hc/\lambda = mc^2$, from which it would appear that a photon has mass $h/\lambda c$ and momentum h/λ. Experimental evidence for this was provided by A. H. Compton,* who observed that in the scattering of X-rays by electrons the law of conservation of momentum is obeyed. In 1924 L. de Broglie † built a bridge between optics and mechanics by suggesting that a particle of mass m and velocity v (momentum p) should have associated with it, like a photon, a wavelength λ determined by the relation

$$\lambda = h/mv = h/p.$$

At first this hypothesis was merely of academic interest, but in 1927 Davisson and Germer ‡ in America and G. P. Thomson § in England observed diffraction effects in experiments on the scattering of electrons by crystal surfaces and by thin metal foils. Thomson showed, moreover, that the wavelengths associated with homogeneous beams of electrons varied with their velocity in accordance with de Broglie's relation.

2. It may, therefore, be assumed that a particle of momentum p and energy E has associated with it a wave function ψ given by

$$\psi = \psi_0 \exp\{2\pi i(x/\lambda - \nu t)\}, \qquad\qquad \text{(iii)}$$

in which λ is the wavelength and ν the frequency.

In atomic problems we are concerned, normally, only with the spatial variation of this function, so that by double differentiation we get the usual equation for "standing" waves:

$$\frac{\partial^2\psi}{\partial x^2} + \left(\frac{2\pi}{\lambda}\right)^2 \psi = 0. \qquad\qquad \text{(iv)}$$

3. For a free particle, using de Broglie's relation, we have

$$E = \tfrac{1}{2}mv^2 = p^2/2m = \tfrac{1}{2}m(h/m\lambda)^2,$$

and the wave equation is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{8\pi^2 mE}{h^2}\psi = 0. \qquad\qquad \text{(v)}$$

The energy E occurs in this equation; it is a one-dimensional form of the Schrödinger equation,‖ which plays in wave mechanics much the same part as that of Newton's equations in classical mechanics.

* A. H. Compton, *Phys. Rev.*, **21**, pp. 207, 483, 715 (1923).

† L. de Broglie, *Phil. Mag.*, p. 446 (1922).

‡ Davisson and Germer, *Phys. Rev.*, **30**, p. 175 (1927).

§ G. P. Thomson, *Proc. Roy. Soc.*, (*A*), p. 378 (1928).

‖ E. Schrödinger, *Collected Papers in Wave Mechanics* (Blackie, 1928).

In general, $\psi = f(x, y, z)$ and (v) assumes the form :

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{8\pi^2 mE}{h^2}\psi = 0,$$

or

$$\nabla^2\psi + \frac{8\pi^2 mE}{h^2}\psi = 0. \qquad \text{(vi)}$$

For a particle in a potential field, account must be taken of the potential energy V, so that if E is the total energy, the wave equation is

$$\nabla^2\psi + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0. \qquad \text{(vii)}$$

This is the usual form of Schrödinger's equation employed in atomic physics. In the solution of a problem the potential energy is substituted in (vii) and the equation is solved for values of ψ which are finite, continuous and single-valued, and the values of the total energy corresponding to these solutions are found.

4. In classical mechanics we have for the energy of a free particle,

$$E = (p_x^2 + p_y^2 + p_z^2)/2m.$$

From equation (vi),

$$E\psi = -(h^2/8\pi^2 m) \cdot \nabla^2\psi.$$

Thus to pass from classical mechanics to wave mechanics we replace the momentum components by $(h/2\pi i)^2 \cdot \nabla^2$ and allow this to operate on the wave function ψ. A similar result is obtained by partial differentiation of equation (iii) with respect to x. We get

$$\frac{\partial\psi}{\partial x} = \frac{2\pi i}{\lambda}\psi = \frac{2\pi i}{h} p_x \psi$$

from which it follows that to the x component of the momentum there belongs the differential operator

$$p_x = \frac{h}{2\pi i}\frac{\partial}{\partial x},$$

similar operators belonging to the y and z components.

Again, differentiation of the wave function with respect to t gives

$$\frac{\partial\psi}{\partial t} = -2\pi i\nu\psi = -2\pi i(E/h)\psi,$$

and for the operator belonging to the energy, we get

$$E = -\frac{h}{2\pi i}\frac{\partial}{\partial t}.$$

We conclude that when the wave function is known we can obtain the components of momentum and the energy, by allowing the operators to act on the appropriate value of ψ.

5. The general method of transformation from classical mechanics to wave mechanics may be summarised as follows. In classical mechanics :

$$H(p, q) - E = 0, \qquad \text{(viii)}$$

in which $H(p, q)$ is the Hamiltonian energy function. In wave mechanics $H(p, q)$ becomes

$$H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right),$$

and applying this and the energy operator to the wave function ψ we get

$$\left\{H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right) + \frac{h}{2\pi i}\frac{\partial}{\partial t}\right\}\psi = 0. \quad \text{.........................(ix)}$$

If, as is usually the case in atomic problems, we require only those solutions in which the wave function consists of an amplitude function independent of the time, and a factor periodic in the time, then

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = \frac{h}{2\pi i}\frac{\partial}{\partial t}\exp\{-2\pi i(E/h)t\} = -E\psi,$$

and (ix) assumes the form

$$\left\{H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q}, q\right) - E\right\}\psi = 0. \quad \text{..........................(x)}$$

To form the particular equation for a given problem, it is only necessary to substitute in (x) the appropriate value of the Hamiltonian function. Thus, for the free electron,

$$H = (p_x^2 + p_y^2 + p_z^2)/2m,$$

and (x) becomes

$$\left\{\frac{1}{2m}\left(\frac{h}{2\pi i}\right)^2\nabla^2 - E\right\}\psi = 0,$$

which is identical with (vi).

For the electron in the hydrogen atom,

$$H = (p_x^2 + p_y^2 + p_z^2)^2/2m - e^2/r,$$

and the wave equation is

$$\left\{\frac{1}{2m}\left(\frac{h}{2\pi i}\right)^2\nabla^2 - \frac{e^2}{r} - E\right\}\psi = 0,$$

or

$$\nabla^2\psi + \frac{8\pi m}{h^2}\left(E + \frac{e^2}{r}\right) = 0.$$

Solutions of this equation for values of ψ which are spherically symmetrical lead to a series of values for the energy E identical with those obtained in the Bohr theory.

6. Finally, it is of interest to note that while in classical mechanics

$$pq - qp = 0,$$

this is not true for the corresponding terms in wave mechanics. In this case the symbols represent differential operators for which the commutative law does not hold. For these,

$$pq - qp = \frac{h}{2\pi i}\left(\frac{\partial}{\partial q}q - q\frac{\partial}{\partial q}\right),$$

and if this operator acts on the wave function ψ, then

$$\frac{h}{2\pi i}\left(\frac{\partial q\psi}{\partial q} - q\frac{\partial\psi}{\partial q}\right) = \frac{h}{2\pi i}\psi,$$

or

$$pq - qp = h/2\pi i.$$

J. A. T.

CORRIGENDA.

Mathematical Gazette, XXXI, No. 295 (July 1947), p. 133, l. 7. *For* $L'C$, read LC; p. 133, last line but one. *For* XK, read K.

A GEOMETRICAL MINIMUM PROBLEM.

By K. J. Le Couteur.

The general problem may be stated as: Given a plane polygon $A_1A_2 \dots A_n$, to find the polygon $A_1' \dots A_n'$ inside $A_1 \dots A_n$ which minimises the perimeter S of $A_1' \dots A_n'$, subject to a condition

$$A_1A_1' + \dots + A_nA_n' = l. \qquad \text{(i)}$$

In the general case some progress is possible, but I have an explicit solution for the cases $n=3$, $n=4$ only.

General case.

Consider small displacements of $A_1', \dots, A_n'$ about the position of minimum perimeter. First suppose A_r' displaced to A_r'', leaving all other A's fixed. Both A_r' and A_r'' satisfy (i), so that $A_rA_r' = A_rA_r''$, and hence in the limit $A_r'A_r''$ is perpendicular to A_rA_r'. The first-order variation in S must also vanish. Hence A_r' and A_r'' lie on an ellipse with foci A'_{r-1}, A'_{r+1}, and in the limit $A_r'A_r''$ is the tangent at A_r' to this ellipse. Then A_rA_r' must be the corresponding normal, and it follows that A_rA_r' is the bisector of the angle $A'_{r-1}A_rA'_{r+1}$.

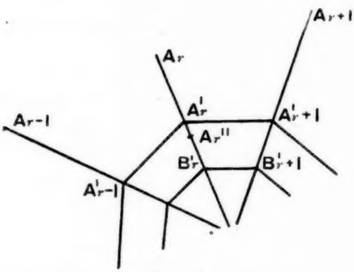

Fig. 1.

Consider now a small displacement of A_r' in the direction of A_rA_r' to A_r'' (Fig. 1). To satisfy (i) this must be associated with similar variations of the other points $A_1', \dots, A_n'$, subject to the condition

$$\sum_r A_r'A_r'' = 0. \qquad \text{(ii)}$$

The first-order variation of the perimeter of $A_1' \dots A_n'$ vanishes if

$$\sum_r A_r'A_r'' \cos \tfrac{1}{2}A_r' = 0, \qquad \text{(iii)}$$

which follows from (ii) if

$$\cos \tfrac{1}{2}A_r' = \cos \tfrac{1}{2}A_1' = \cos \tfrac{1}{2}A_n'. \qquad \text{(iv)}$$

Combining these results, we have:

Theorem I. The polygon $A_1' \dots A_n'$ which, subject to (i), *has minimum perimeter is equiangular and, for each r, A_rA_r' is the bisector of the angle at its vertex A_r'.*

These results are otherwise apparent. Consider a loop of inextensible string which passes over pulleys at the points $A_1, \dots, A_n$ and is drawn tight by an elastic band $A_1' \dots A_n'$. There is no friction and the position of static equilibrium is that of minimum length of the elastic band, so that this mechanical system solves our problem, the value of l being determined by the length of

the inelastic string. This method is applicable also if $A_1, \dots, A_n$ are not coplanar. It is now obvious that A_rA_r' must bisect $\angle A_r'$.

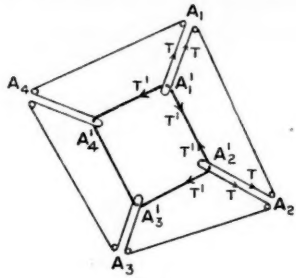

FIG. 2.

For equilibrium we also require, resolving in A_rA_r',

$$T = T' \cos \tfrac{1}{2}A_r,$$

which is just equation (iv). The previous treatment corresponds to a solution of the problem by the method of virtual work.

Theorem II. Given any solution $A_r', \dots, A_n'$ of the problem, we obtain a related solution $B_1' \dots, B_n'$ by taking points B_r' on A_rA_r' such that

$$A_1'B_1' = A_r'B_r' = x, \text{ say (Fig. 1)};$$

then
$$A_1B_1' + \dots + A_nB_n' = l + nx.$$

This construction may be used to determine the solution for a given value of l if the solution for any other value is known.

The procedure of Theorem II gives rise to polygons $B_1', \dots, B_n'$ lying wholly or partially outside $A_1 \dots A_n$. By adopting a suitable sign convention for the distances A_rB_r' such polygons can be regarded as solutions of the problem, and the corresponding values of l may be zero or negative. It is possible to devise a more complicated mechanical model which treats all cases, but as this generalisation is rather forced, I give no details.

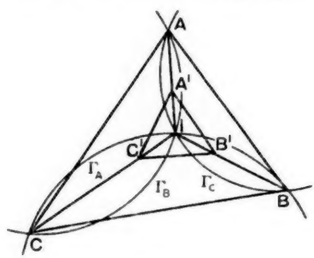

FIG. 3.

A related problem.

Any solution $A_1' \dots A_n'$ satisfying Theorem I also solves the following problem: Given $A_1 \dots A_n$, to find the polygon $A_1' \dots A_n'$ of given perimeter S for which $A_1A_1' + \dots + A_nA_n'$ has a turning value.

Case $n=3$.

A, B, C are given. Theorem I shows that $A'B'C'$ is equilateral and that AA', BB', CC' meet in the incentre I of $A'B'C'$ (Fig. 3). Hence

$$\angle BIC = \angle CIA = \angle AIB = 2\pi/3,$$

and so I may be constructed as the intersection of three circles Γ_a, Γ_b, Γ_c. Theorem II then defines the solution for any particular value of l. I is itself a degenerate triangle, and so according to the remark above minimises $AI+BI+CI$.

Special consideration is necessary if one of the angles of ABC is greater than $2\pi/3$. Suppose $\angle A > 2\pi/3$ (Fig. 4). Then the circles Γ_a, Γ_b, Γ_c meet in

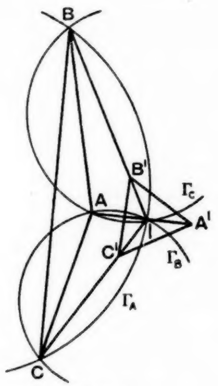

FIG. 4.

a point I exterior to ABC such that $\angle BIC = 2\pi/3$, $\angle CIA = \angle AIB = \pi/3$. Again, I is the incentre of a triangle $A'B'C'$ which is a formal solution of the problem if AA' is regarded as negative and BB', CC' as positive.

However, in this special case, if $A'B'C'$ is restricted to lie within ABC, then the triangle of least perimeter is obtained by taking A' at A. Our previous discussion is still applicable to the variation of B' and C' so that $AB'C'$ is isosceles with BB', CC', bisecting the angles B' and C'. We have (Fig. 5):

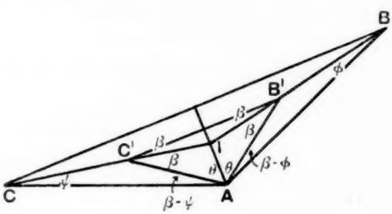

FIG. 5.

$$BB' : B'A : BA = \sin(\beta-\phi) : \sin\phi : \sin\beta,$$
$$CC' : C'A : CA = \sin(\beta-\psi) : \sin\psi : \sin\beta.$$

Thus, since $B'A = C'A$,

$$BA \sin \phi = CA \sin \psi = r, \text{ say}. \qquad \text{(v)}$$

Thus BI, CI are tangents to a circle, centre A and radius r, and for any given r are easily constructed. The location of $B'C'$ is then trivial, for β being now known, θ is determined by the equation

$$\theta + 2\beta = \tfrac{1}{2}\pi.$$

There is no simple relationship between r and l. However, given r, (v) determines ϕ and ψ; then β is determined by the equation

$$A = \pi - 2\beta - \phi - \psi$$

and l by

$$BB' + CC' = l = [BA \sin (\beta - \phi) + CA \sin (\beta - \psi)]/\sin \beta.$$

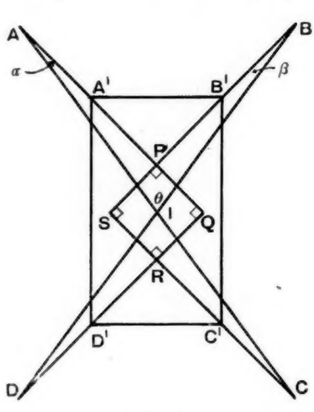

Fig. 6.

Case $n = 4$.

This is more complicated than $n = 3$, because $A'B'C'D'$, though rectangular, is not necessarily square, so that the bisectors AA', BB', CC', DD' are not concurrent, but rather define a square $PQRS$ (Fig. 6).

Given $PQRS$, then starting from any point A' on AP we may complete the rectangle $A'B'C'D'$ by choosing B', C', D' so that

$$PA' = PB' = RC' = RD',$$

and the rectangle may be adjusted to any required l by the method of Theorem II.

The problem is now reduced to the construction of $PQRS$. AC and BD meet in I, and without loss of generality we may suppose

$$\angle AIB = \theta < \tfrac{1}{2}\pi.$$

Considering $APBI$, we have

$$\alpha + \beta + 3\pi/2 + \theta = 2\pi,$$

or

$$\alpha + \beta = \tfrac{1}{2}\pi - \theta = \lambda, \text{ say}. \qquad \text{(vi)}$$

We require also $QR = PQ$, or

$$AC \sin \alpha = BD \sin \beta. \qquad \text{(vii)}$$

Now, λ, AC, BD being known, (vi) and (vii) are easily solved.

Construct a triangle OXY (Fig. 7), such that

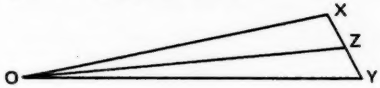

FIG. 7.

$$OX=AC,\ OY=BD,\ \angle XOY=\lambda;$$

then if Z is the midpoint of XY,

$$OX \sin XOZ = OY \sin YOZ = XZ \sin XZO,$$

so that $\angle XOZ=\alpha$, $\angle YOZ=\beta$ is a solution of (vi) and (vii). Given α and β, $PQRS$ may be constructed and the problem is solved.

In the special case $\theta=\frac{1}{2}\pi$, P, Q, R, S coincide at I.

This problem arises out of a railway journey undertaken by Mr. Goddard, who wishing to visit A, B, C (Oxford, Cambridge, London) and being unwilling to take three single railway tickets used instead three return tickets for the journeys AA', BB', CC' and cycled the distance $A' \to B' \to C' \to A'$.

K. J. LE COUTEUR.

NACHRICHTEN DER MATHEMATISCHEN GESELLSCHAFT IN WIEN.

THE President of the Vienna Mathematical Society, Professor Rudolf Inzinger, in sending the Association a copy of the *Nachrichten*, informs us that the Society was re-established in 1945. The *Nachrichten*, to be issued three times a year, will, it is hoped, prove a contribution of real importance to the furtherance of international collaboration in the mathematical field.

Mathematicians in this country who would be glad to receive copies, free of charge, are requested to write at once to

Dr. Rudolf Inzinger, Karlplatz 13, Wien IV, Austria.

EUREKA.

OLD friends of *Eureka*, who welcome amusement as well as edification in their mathematical pursuits, will be glad to learn that a new number is in preparation, and that it should be on sale early in 1948. Those who do not know *Eureka* should remedy this defect promptly, for the journal of the Archimedeans, the undergraduate mathematical society of the University of Cambridge, is always interesting, instructive and amusing. The Archimedeans will naturally welcome a large demand for the new number, and the support of members of the Mathematical Association should be heartily accorded to the junior body. The price of the new number is 2s., post free. Orders, with the money, should be sent at once to

The Editor (G. C. Shephard), Queens' College, Cambridge.

The new number is No. 10. A few copies of Nos. 8 and 9 are still available at 1s. and 1s. 6d. each respectively.

DISTRIBUTION OF THE OCTOBER *GAZETTE*.

OWING to a mishap in the distribution, it is believed that the October *Gazette* may not have been sent to certain members. If members who did not receive a copy of the October issue will kindly notify Mr. Parsons, Merchant Taylors' School, Sandy Lodge, Northwood, Middlesex, every effort will be made to supply the missing number, provided that application is made immediately.

SOME PROPERTIES OF THE NINE-POINTS CIRCLE.

By E. A. Maxwell.

Prologue. This article began in an attempt, for purposes which shall be nameless, to find the result of inverting a triangle and associated lines and circles with respect to its nine-points circle. The work outran its aim, and I now present it from a quite different starting-point.

1. If
$$S \equiv a(x^2+y^2)+2gx+2fy+c,$$
then the equation $S=0$, in rectangular cartesian coordinates, represents a circle which we can conveniently denote by the same letter S. We shall, further, use the symbol **S** to denote the entity (a, g, f, c) which may be regarded as defining the homogeneous coordinates of a point in space of three dimensions.*

If S, S' are two circles, whose symbols are $\mathbf{S}$, $\mathbf{S}'$, then the symbol $\mathbf{S}+k\mathbf{S}'$ defines, as the parameter k varies, the circles of a *pencil* or, as it is called, a *coaxal system*. Our use of the word " circle " includes as a special case the straight line, which arises when the coefficient a vanishes. In particular, if S and S' are *both* straight lines, then the symbol $\mathbf{S}+k\mathbf{S}'$ defines a pencil of straight lines in the ordinary sense of the term.

2. Let ABC be a triangle which, for convenience, we assume to be acute-angled and scalene. We denote the altitudes by AP, BQ, CR, meeting in the orthocentre H; and we take the middle points of BC, CA, AB, AH, BH, CH to be D, E, F, U, V, W respectively. It will be convenient to give here a list of the names which we shall give to certain circles (or straight lines) which occur frequently :

$$\begin{array}{rrr} BCDP\equiv a, & CAEQ\equiv b, & ABFR\equiv c;\\ AHP\equiv p, & BHQ\equiv q, & CHR\equiv r;\\ QR\equiv p', & RP\equiv q', & PQ\equiv r';\\ AQHR\equiv \alpha, & BRHP\equiv \beta, & CPHQ\equiv \gamma;\\ BCQR\equiv \alpha', & CARP\equiv \beta', & ABPQ\equiv \gamma';\\ ABC\equiv t; & & PQRDEFUVW\equiv \delta. \end{array}$$

The corresponding letter in **clarendon** type will give the symbol whose elements are the coefficients in the equation of the circle, as described in § 1.

3. In order to set up our " coordinate system " of symbols, we need four basic symbols in terms of which all the others can be expressed. We shall take these † to be $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$. Moreover, we shall assume that multiples are taken so that for each of them the coefficient of x^2 and y^2 (which is not zero) has the value unity. We now determine the symbols of the straight lines and circles listed in § 2.

4. The line p belongs to the pencil defined by β and γ or, as we shall say, to the pencil (β, γ). Hence $\mathbf{p}$ is of the form $k'\boldsymbol{\beta}+k''\boldsymbol{\gamma}$, where the constants are to be chosen so that the coefficient of x^2 and y^2 vanishes. Similar results hold for q, r, p', q', r', and so we obtain their symbols as follows :

$$\mathbf{p}\equiv\boldsymbol{\beta}-\boldsymbol{\gamma},\quad \mathbf{q}\equiv\boldsymbol{\gamma}-\boldsymbol{\alpha},\quad \mathbf{r}\equiv\boldsymbol{\alpha}-\boldsymbol{\beta};$$
$$\mathbf{p}'\equiv\boldsymbol{\delta}-\boldsymbol{\alpha},\quad \mathbf{q}'\equiv\boldsymbol{\delta}-\boldsymbol{\beta},\quad \mathbf{r}'\equiv\boldsymbol{\delta}-\boldsymbol{\gamma}.$$

* See an article by the author on pp. 46–9, and by Dr. D. Pedoe on pp. 210–15, of *Math. Gazette*, vol. XXI (1937).

† Note that the whole configuration is determined (in two ways) if the circles β, γ are given.

Using these results, we are able to find the symbols $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. For a is the harmonic conjugate of p with respect to q', r', and

$$-\mathbf{p} \equiv \mathbf{q}' - \mathbf{r}'.$$

Hence $\mathbf{a}$ is a multiple of $\mathbf{q}' + \mathbf{r}'$, and we take

$$\mathbf{a} \equiv 2\boldsymbol{\delta} - \boldsymbol{\beta} - \boldsymbol{\gamma}, \quad \mathbf{b} \equiv 2\boldsymbol{\delta} - \boldsymbol{\gamma} - \boldsymbol{\alpha}, \quad \mathbf{c} \equiv 2\boldsymbol{\delta} - \boldsymbol{\alpha} - \boldsymbol{\beta}.$$

To determine $\boldsymbol{\alpha}'$, we note that the circle α' belongs at once to the pencil (b, γ) and to the pencil (c, β). Hence we have the relations :

$$\boldsymbol{\alpha}' \equiv 2\boldsymbol{\delta} - \boldsymbol{\alpha}, \quad \boldsymbol{\beta}' \equiv 2\boldsymbol{\delta} - \boldsymbol{\beta}, \quad \boldsymbol{\gamma}' \equiv 2\boldsymbol{\delta} - \boldsymbol{\gamma}.$$

Finally, the circumcircle t belongs to each of the pencils (a, α'), (b, β'), (c, γ'), and so

$$\mathbf{t} \equiv 4\boldsymbol{\delta} - \boldsymbol{\alpha} - \boldsymbol{\beta} - \boldsymbol{\gamma}.$$

These symbols are capable of an interesting interpretation in space after the manner described in § 1. If we call $\alpha^\star$ the point in space which corresponds to the circle α, with similar notation for the other letters, then the relations $\boldsymbol{\alpha}' \equiv 2\boldsymbol{\delta} - \boldsymbol{\alpha}$, etc., show that the triangles $\alpha^\star\beta^\star\gamma^\star$ and $\alpha'^\star\beta'^\star\gamma'^\star$ are in perspective from $\delta^\star$. Moreover, the axis of perspective contains the points whose symbols are $\boldsymbol{\beta}' - \boldsymbol{\gamma}' \equiv -(\boldsymbol{\beta} - \boldsymbol{\gamma})$, etc., and so the axis of perspective is the line $p^\star q^\star r^\star$. Hence we have the following theorem :

The circles α, β, γ ; α', β', γ' are represented in space by triangles in perspective from the point which represents δ, the axis of perspective being the line on which lie the points representing the altitudes p, q, r.

5. There are one or two other circles whose symbols are useful for subsequent work. We consider first the circle HPD. It belongs to both of the pencils (a, δ), (β, γ), and so its symbol is $\boldsymbol{\beta} + \boldsymbol{\gamma}$. We have therefore the three symbols :

$$HPD, \boldsymbol{\beta} + \boldsymbol{\gamma} \; ; \quad HQE, \boldsymbol{\gamma} + \boldsymbol{\alpha} \; ; \quad HRF, \boldsymbol{\alpha} + \boldsymbol{\beta}.$$

Considerations of symmetry suggest that we should now interpret the symbol $\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma} \equiv \boldsymbol{\rho}$, say. Writing this symbol in the form $\boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma})$, we see that the circle ρ passes through the points common to the circles $AHQR$ and HPD ; one of these points is H, and the other is easily identified as the foot of the perpendicular from H to the median AD. Similar results follow for the medians BE, CF, and so ρ *is the circle on GH as diameter,* where G is the centroid of the triangle ABC.

6. Certain results can now be deduced quickly. We give one or two examples :

(i) The identity $\mathbf{t} \equiv 4\boldsymbol{\delta} - \boldsymbol{\rho}$ shows that *the circumcircle, the nine-points circle and the circle on GH as diameter are coaxal.*

(ii) The radical axis of the circumcircle and the nine-points circle is given by the symbol $3\boldsymbol{\delta} - \boldsymbol{\alpha} - \boldsymbol{\beta} - \boldsymbol{\gamma}$ (since the coefficient of x^2 and y^2 vanishes). We can write this in the alternative forms $\mathbf{a} + \mathbf{p}' \equiv \mathbf{b} + \mathbf{q}' \equiv \mathbf{c} + \mathbf{r}'$, from which it follows, after one or two easy steps, that *the radical axis of the circumcircle and the nine-points circle is the polar line of the orthocentre with respect to the given triangle.*

(iii) The circle on the median AD as diameter belongs to each of the pencils (a, δ), (β', γ'), and so its symbol is $4\boldsymbol{\delta} - \boldsymbol{\beta} - \boldsymbol{\gamma}$. We thus have the symbols :

$$ADP, 4\boldsymbol{\delta} - \boldsymbol{\beta} - \boldsymbol{\gamma} \; ; \quad BEQ, 4\boldsymbol{\delta} - \boldsymbol{\gamma} - \boldsymbol{\alpha} \; ; \quad CFR, 4\boldsymbol{\delta} - \boldsymbol{\alpha} - \boldsymbol{\beta}.$$

Hence *the circles which have the medians BE and CF as diameters intersect on the altitude AP.*

(iv) The circles t, α and the circle on AD as diameter are coaxal, meeting in two points, of which one is A. Hence *the line HD meets the circumcircle again in two points, one of which is the opposite end of the diameter through A* (and the other of which is the foot of the perpendicular from A to HD).

Interlude. I developed the technique described in the preceding paragraphs to verify and perhaps to extend some results found by inverting a triangle with respect to its nine-points circle. I found, however, that I was unable to solve the following problems, any one of which would have settled the others : (i) to express the symbols of the medians ; (ii) to express the symbols of the circles such as ADB, ADC ; (iii) to express the symbol of the Euler line. I think that the actual coefficients in the basic circles are required, equivalent perhaps to the lengths of the sides of the triangle ABC. At any rate, I was unable to pin down certain results to the nine-points circle, and I found with some surprise that they were actually more general than appeared at first sight. These results I now describe.

7. Let K be an arbitrary point of the plane (originally meant to be the centre of the nine-points circle). Let the point of intersection, other than K, of the circles KQE, KRF be A', with similar notation for B' and C'. We denote these circles by the names θ, ϕ, ψ, so that

$$\theta \equiv KPDB'C', \quad \phi \equiv KQEC'A', \quad \psi \equiv KRFA'B'.$$

Since θ belongs to the coaxal system (a, δ), $\boldsymbol{\theta}$ can be expressed in the form :

$$\boldsymbol{\theta} \equiv \lambda\boldsymbol{\delta} + \boldsymbol{\beta} + \boldsymbol{\gamma}.$$

Similarly,

$$\boldsymbol{\varphi} \equiv \mu\boldsymbol{\delta} + \boldsymbol{\gamma} + \boldsymbol{\alpha},$$

$$\boldsymbol{\psi} \equiv \nu\boldsymbol{\delta} + \boldsymbol{\alpha} + \boldsymbol{\beta}.$$

Now

$$\boldsymbol{\varphi} - \boldsymbol{\psi} \equiv (\mu - \nu)\boldsymbol{\delta} - (\boldsymbol{\beta} - \boldsymbol{\gamma})$$

$$\equiv (\mu - \nu)\boldsymbol{\delta} - \mathbf{p}.$$

But $\boldsymbol{\varphi} - \boldsymbol{\psi}$ is the symbol of a circle through K, A', and $(\mu - \nu)\boldsymbol{\delta} - \mathbf{p}$ is the symbol of a circle through P, U, so that the points K, A', P, U are concyclic. Hence, *whatever the position of K, the circle KPA′ passes through U.* Similarly, the circle KQB' passes through V and the circle KRC' through W.

Moreover,

$$(\boldsymbol{\varphi} - \boldsymbol{\psi}) + (\boldsymbol{\psi} - \boldsymbol{\theta}) + (\boldsymbol{\theta} - \boldsymbol{\varphi}) \equiv 0,$$

and so the three circles $KPA'U$, $KQB'V$, $KRC'W$ are coaxal. They all pass through K and another point, say H'. Hence, *if K is an arbitrary point of the plane, the circles KPU, KQV, KRW have a further common point H′.*

We now prove that A' lies on the straight line AK. In fact, the radical axis $A'K$ of the circles ϕ, ψ is given by the symbol (whose coefficient of x^2 and y^2 vanishes) :

$$(\nu + 2)\boldsymbol{\varphi} - (\mu + 2)\boldsymbol{\psi}.$$

But

$$\boldsymbol{\varphi} \equiv (\mu + 2)\boldsymbol{\delta} - \mathbf{b}, \quad \boldsymbol{\psi} \equiv (\nu + 2)\boldsymbol{\delta} - \mathbf{c},$$

and so the symbol of $A'K$ is

$$-(\nu + 2)\mathbf{b} + (\mu + 2)\mathbf{c},$$

which is the symbol of a line through A, as required. Hence we have the result that *KA′A, KB′B, KC′C are straight lines.*

We next show that *the straight line KH′ passes through the orthocentre H for all positions of K.* In fact, the points K, H' lie on the two circles $KQVH'$, $KRWH'$ whose symbols are $(\nu - \lambda)\boldsymbol{\delta} - \mathbf{q}$, $(\lambda - \mu)\boldsymbol{\delta} - \mathbf{r}$, as before. The radical

axis KH' of these two circles is given by the symbol (whose coefficient of x^2 and y^2 vanishes):

$$(\nu-\lambda)\{(\lambda-\mu)\boldsymbol{\delta}-\mathbf{r}\}-(\lambda-\mu)\{(\nu-\lambda)\boldsymbol{\delta}-\mathbf{q}\}$$
$$\equiv(\lambda-\mu)\mathbf{q}-(\nu-\lambda)\mathbf{r},$$

which is the symbol of a straight line through H, as required.

Note that the symbol of the line HKH' can be expressed more symmetrically in the form:

$$(\lambda-\mu)(\boldsymbol{\gamma}-\boldsymbol{\alpha})-(\nu-\lambda)(\boldsymbol{\alpha}-\boldsymbol{\beta})$$
$$\equiv(\mu-\nu)\boldsymbol{\alpha}+(\nu-\lambda)\boldsymbol{\beta}+(\lambda-\mu)\boldsymbol{\gamma}.$$

8. The use of the symbols made the properties given in § 7 easy to "spot", but once they are noticed it is not hard to prove them by straightforward methods. It will both indicate a neat method of proof and express the results in more symmetrical form if we give the following definition of the points A', B', C', H':

The points A', B', C', H' are respectively the inverses of K with respect to the circles whose centres are A, B, C, H and the squares of whose radii are the powers of those points with respect to the nine-points circle.

For an acute-angled triangle ABC the power of H with respect to the nine-points circle is negative. This merely means that H, H' are on opposite sides of K.

This result leads us naturally to enquire whether there are positions of K for which (A, A'), (B, B'), (C, C'), (H, H') are themselves inverse points with respect to a circle whose centre is K. If so, the circle $KB'C'$ would invert into the line BC, and the points D, P in which the circle meets the line would invert into themselves. In like manner, the points E, Q and F, R would invert into themselves. The required circle of inversion does therefore exist; it is the nine-points circle, and we arrive back at the point from which this note arose.

9. The properties derived from inversion with respect to the nine-points circle form an excellent illustration of work which it is better fun to derive for oneself than to read about. The results of §§ 7–8 give a foundation from which the configuration can be determined, and further detail may perhaps be omitted. One point might, however, be emphasised: the inversion is with respect to the circle itself, and not merely with respect to *any* circle whose centre is at the nine-points centre; hence the inverted figure is, so to speak, superposed on the original triangle, giving us further properties of that triangle.

Epilogue. I ought to add that I cannot believe that much of the above work is new, but I am unable to find it in what are to me the standard textbooks. Anyway, it does not seem to have been done recently, and a new generation may like to be reminded of the figure. Obviously there are innumerable points where fresh investigations are suggested, and that is my chief excuse for writing. E. A. M.

GLEANINGS FAR AND NEAR.

1551. The face of the man Müller speaks to me in the old and long-written language of human expression. It is a terrible face and full of evil, full of logic and subtlety and craft. It is the face of a mathematician, yet the face of a satyr. It is cold as ice.—H. De Vere Stacpoole, *The Cottage on the Fells*, p. 144. [Per Mr. S. Thomson.]

1552. At Llyn Gwyn I watched the portentous strides of two herons. The way they walked one might think that their heads were weighed down with a problem in higher mathematics, instead of being just occupied with the idea of finding a frog or an eel.—R. Gibbings, *Coming down the Wye* (Dent), Ch. 9. [Per Mr. I. FitzRoy Jones.]

ON CERTAIN CONFIGURATIONS OF CONGRUENT TRIANGLES.

By D. G. Taylor.

1. It is proposed to consider all triangles congruent with a given triangle, and having their vertices on the sides (or sides produced) of the given one, with the various alternative allocations of vertices and sides; and, conversely, all triangles congruent to a given one, but whose sides (or sides produced) pass through the vertices of the given one, again with the various allocations.

A few preliminary theorems and constructions, more or less familiar, must be tabulated for reference.

2. *An identical relation between the angles of two triangles.*

Consider the expression:

$$\sin^2 B \sin^2 Z + \sin^2 C \sin^2 Y - 2 \sin B \sin C \sin Y \sin Z \cos (A + X),$$

where $A + B + C = \pi$, $X + Y + Z = \pi$. This expression is, by its form, positive, and we shall denote it by P^2. Expanding $\cos (A + X)$, and using the identities

$$\cos A \sin B \sin C = \tfrac{1}{2}(- \sin^2 A + \sin^2 B + \sin^2 C)$$

and the like for X, Y, Z, we have

$$P^2 = \tfrac{1}{2}(- \sin^2 A + \sin^2 B + \sin^2 C) \sin^2 X + \ldots + 2\Pi (\sin A)\Pi (\sin X),$$

which remains unchanged when A, B, C are respectively interchanged with X, Y, Z. Hence

$$P^2 = \Sigma (\cos A \sin B \sin C \sin^2 X) + 2\Pi (\sin A)\Pi (\sin X), \quad \text{.....................(i)}$$
$$= \Sigma (\cos X \sin Y \sin Z \sin^2 A) + 2\Pi (\sin X)\Pi (\sin A).$$

When X, Y, Z become equal to A, B, C respectively, P reduces to the value 2Π $(\sin A)$.

Thus P remains unchanged when, simultaneously, A is interchanged with X, B with Y, and C with Z; and also when any of the pairs A, X; B, Y; C, Z; is interchanged with another pair.

3. *Pedal and antipedal triangles of a point.*

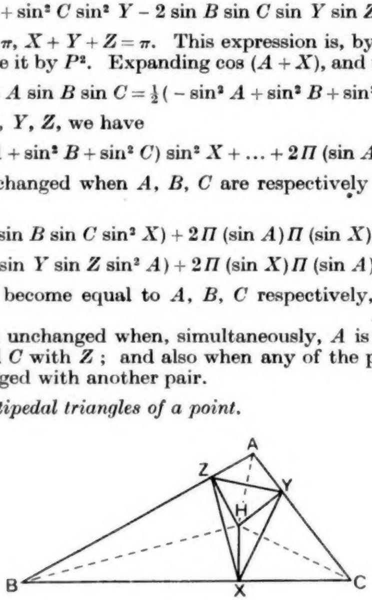

Fig. 1.

(A) Given (Fig. 1) a triangle ABC, and a second triangle (not shown in the figure), with any angles X, Y, Z. Find H, the point at which BC, CA, AB subtend the angles $A + X$, $B + Y$, $C + Z$; drop perpendiculars HX, HY, HZ to BC, CA, AB. Then XYZ (the *pedal* triangle of H with respect to ABC) is the *minimum* triangle of this shape inscribable in ABC, with this allocation of vertices to sides. Its angles are clearly equal to the given X, Y, Z respectively; and that it is the minimum is clear from the fact that if we draw HX', HY', HZ' making (say) the positive angle α with HX, HY, HZ respectively, we obtain a triangle $X'Y'Z'$ similar to XYZ, but increased in linear

dimensions in the ratio sec $\alpha : 1$; and likewise when α is measured in the other sense.

(B) With the same data as in (A), find K (Fig. 2), the point at which BC, CA, AB subtend the angles $\pi - X$, $\pi - Y$, $\pi - Z$ respectively. Join KA, KB, KC; perpendicular to these draw Y_2AZ_2, Z_2BX_2, X_2CY_2 to form a triangle $X_2Y_2Z_2$ circumscribed to ABC. Then $X_2Y_2Z_2$ (the *antipedal* triangle of K with respect to ABC) is the *maximum* triangle of this shape circumscribable about ABC. Its angles are clearly equal to X, Y, Z respectively; and it is easily seen to be the maximum by drawing the circular arcs which are the loci of the vertices for circumscribed triangles of this shape.

We may refer to the above two constructions as " construction (A) " and " construction (B) ".

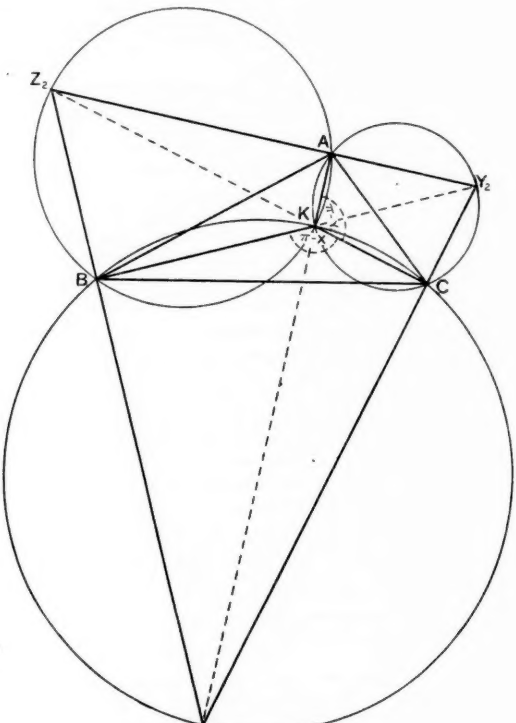

FIG. 2.

4. Let the triangles ABC, XYZ, $X_2Y_2Z_2$ be for short denoted by Δ, Δ_1, Δ_2, and their circumradii by R, R_1, R_2. We obtain expressions for R_1, R_2.

We have (Fig. 1) from the cyclic quadrilaterals of type $AZHY$,

$$AH \,.\, \sin A = x (\equiv YZ), \quad x = 2R_1 \sin X, \text{ etc.};$$

and from the triangle BHC

$$a^2 = BH^2 + HC^2 - 2BH \,.\, HC \cos BHC,$$

which leads to $$4R^2 \sin^2 A \sin^2 B \sin^2 C = 4R_1^2 P^2,$$

or $$R_1 = R\Pi (\sin A)/P. \quad \text{(ii)}$$

Also (Fig. 2), since ABC is the *pedal* triangle of K with respect to $X_2Y_2Z_2$,

$$R = R_2 \Pi (\sin X)/P,$$

or $$R_2 = PR/\Pi (\sin X). \quad \text{(iii)}$$

From (ii) and (iii),

$$R_1R_2 = R^2 \Pi (\sin A)/\Pi (\sin X), \quad \text{(iv)}$$

and $$R_2/R_1 = P^2/\Pi (\sin A) \Pi (\sin X). \quad \text{(v)}$$

5. *The sides of $X_2Y_2Z_2$ are respectively parallel to those of XYZ.*

To prove this we find expressions for $\cos AYZ$, $\cos Y_2AC$.

We have (Fig. 1),

$$\cos AYZ = \cos AHZ = HZ/AH\,;$$

from the triangle AHB

$$c \,.\, HZ = AH \,.\, HB \sin AHB.$$

Hence $$\cos AYZ = 2R_1 \sin Y \sin (C+Z)/2R \sin B \sin C$$
$$= \sin A \sin Y \sin (C+Z)/P. \quad \text{(vi)}$$

Also in Fig. 2,

$$\cos Y_2AC = \cos Y_2KC = KC/KY_2.$$

But in the triangle X_2KY_2,

$$X_2Y_2 \,.\, KC = KX_2 \,.\, KY_2 \sin X_2KY_2.$$

Thus $$\cos Y_2AC = KX_2 \sin X_2KY_2/X_2Y_2$$
$$= 2R \sin A \sin (C+Z)/2R_2 \sin X \sin Z$$
$$= \sin A \sin Y \sin (C+Z)/P. \quad \text{(vii)}$$

Now (vi) and (vii) show that Y_2Z_2 is parallel to YZ; similarly for the other sides. This means, of course, that AK is perpendicular to YZ, and so for the others, giving a direct construction for K, when H is given. A similar process proves that the angles KAB, HAC are equal, and so at each vertex; that is, K, H are *isogonal conjugates*.

6. Suppose we are given, in addition to ABC, a triangle $X_0Y_0Z_0$ with angles X, Y, Z, and greater than the minimum size XYZ of Fig. 1. Then by rotation from XYZ through the proper angle α, positively and negatively about H, we obtain two triangles *congruent* to $X_0Y_0Z_0$, and having their vertices on BC, CA, AB respectively. If we vary the allocation of vertices to sides, there are six permutations of the angles X, Y, Z, yielding *twelve* such triangles.

Similarly, if $X_0Y_0Z_0$ is less than the maximum size $X_2Y_2Z_2$ of Fig. 2, then by rotation through the proper angle, positively or negatively, about K, we obtain two triangles *congruent* to $X_0Y_0Z_0$, and having their sides passing through A, B, C; and, here also, twelve such triangles are obtainable by permuting the vertices.

7. Now if the triangles ABC, $X_0Y_0Z_0$ are *similar*, in *any* order of vertices, $\Pi (\sin X) = \Pi (\sin A)$, and between the circumradii of the minimum and maximum triangles XYZ, $X_2Y_2Z_2$ we have the relation

$$R_1R_2 = R^2. \quad \text{(viii)}$$

We can then choose such an angle α of rotation, positively or negatively about H, as to obtain two triangles $X'Y'Z'$, $X''Y''Z''$, *congruent* to ABC and having their vertices on BC, CA, AB, in this particular allocation. We can also choose such an angle β of rotation, positively or negatively about K, as to obtain two triangles $X_2'Y_2'Z_2'$, $X_2''Y_2''Z_2''$, *congruent* to ABC and having their sides passing through A, B, C, in this particular allocation.

It will now be proved that $\beta=\alpha$, so that the two latter triangles are similarly situated to the two former.

The rotation α increases the linear dimensions of XYZ in the ratio sec $\alpha : 1$, so that the circumradius of the new triangle is, say, $R_1'=R_1 \sec \alpha$.

From Fig. 2 it is clear that the rotation β *reduces* the linear dimensions of $X_2Y_2Z_2$ in the ratio $\cos \beta : 1$, so that the circumradius of the new triangle is, say, $R_2'=R_2 \cos \beta$. Then

$$R_1'R_2'=R_1R_2 \sec \alpha \cos \beta = R^2 \sec \alpha \cos \beta.$$

Now make $R_1'=R_2'=R$, and it follows that $\beta=\pm\alpha$.

We shall then have the following situation: *of the four triangles $X'Y'Z'$, $X''Y''Z''$, $X_2'Y_2'Z_2'$, $X_2''Y_2''Z_2''$, all are congruent to ABC, in a particular order; the vertices of the two former lie respectively on the sides BC, CA, AB; the sides of the two latter pass respectively through the vertices A, B, C; and the sides of the two latter are respectively parallel to the sides of the two former.*

By permutation there are six ways in which this can happen, yielding six pairs of triangles of each kind (that is, twelve inscribed and twelve circumscribed). But it will be shown later that, in each of two cases, one triangle of the pair coincides with ABC, reducing the twelve to ten. Thus in the general case there are *ten* solutions of each of the two problems posed in § 1.

8. *The six cases.*

Of the six cases, we shall find that one stands alone in simplicity and elegance; that two others call for combined treatment; while the remaining three have no obvious notable qualities. But common to each is what may be called the phenomenon of the *five triangle configuration* described in § 7.

The ten triangles associated with construction (A), which may be called by a stretch of language the ten " inscribed " triangles, will be marked $A_rB_rC_r$ ($r=1, 2, \dots, 10$), while the ten associated with construction (B), the " circumscribed " triangles, will be indicated by accented letters $A_r'B_r'C_r'$.

We shall deal with the cases in the following order:

(α) *Case* I. $X_0Y_0Z_0$ congruent to ABC (in this order).

(β) *Case* II. $X_0Y_0Z_0$ congruent to CAB,

Case III. $X_0Y_0Z_0$ congruent to BCA.

(γ) *Case* IV. $X_0Y_0Z_0$ congruent to ACB,

Case V. $X_0Y_0Z_0$ congruent to CBA,

Case VI. $X_0Y_0Z_0$ congruent to BAC.

9. (α) *Case* I.

In this case H coincides with O, the circumcentre of ABC, and K with the orthocentre P. Since the minimum triangle XYZ has half the linear dimensions of ABC, the rotation necessary for congruence is $\alpha=\pm\pi/3$. The two triangles are marked $A_1B_1C_1$, $A_2B_2C_2$ (Fig. 3). Clearly the former can be obtained from ABC by rotation through $+2\pi/3$ about the point O_+, in which the right bisectors of AA_1, BB_1, CC_1 concur; while the latter comes by rotation of ABC through $-2\pi/3$ about the corresponding point O_-. These points will shortly be identified.

The notable feature in this case is disclosed if we subject $A_1B_1C_1$ to a further rotation $+2\pi/3$ about O_+, to produce a new triangle $A_1'B_1'C_1'$. The three triangles $\triangle$, $\triangle_1$, $\triangle_1'$ have a *cyclic relation*, the vertices of each lying on the sides of the preceding one in the cycle. It will be noticed that, in the figure, $B_1'C_1'$, $C_1'A_1'$, $A_1'B_1'$ pass respectively through A, B, C.

Similarly a rotation of $A_2B_2C_2$ through $-2\pi/3$ about O_- produces $A_2'B_2'C_2'$, completing the cyclic triad ABC, $A_2B_2C_2$, $A_2'B_2'C_2'$; the sides of $A_2B_2C_2$ being parallel to those of $A_1'B_1'C_1'$, and those of $A_2'B_2'C_2'$ to those of $A_1B_1C_1$.

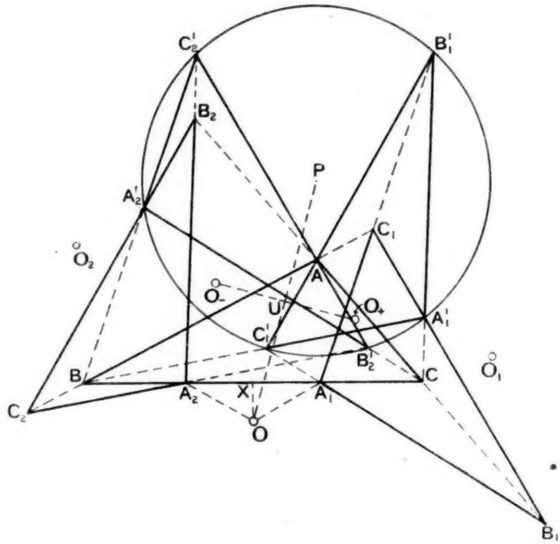

Fig. 3.

10. These triangles have been obtained by a first and then a second application of construction (A). Since construction (B) amounts to the *reversion* of construction (A), which is cyclic of order three, direct application of (B) to ABC leads to $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$. Several points now become clear:

(i) $A_1'B_1'C_1'$, $A_2'B_2'C_2'$ have a common circumcircle, whose centre is P, the orthocentre of ABC.

(ii) O, O_1, P, the circumcentres of $\triangle$, $\triangle_1$, $\triangle_1'$, are equidistant on the circle OO_1PO_-, centre O_+; O, O_2, P, the circumcentres of $\triangle$, $\triangle_2$, $\triangle_2'$, are likewise equidistant on the circle OO_2PO_+, centre O_-.

(iii) OP, O_+O_- bisect each other at right angles at a point U, which is clearly the nine-points centre of ABC.

(iv) OO_-PO_+ are the vertices of a "diamond" or 60° rhombus. Thus O_+, O_- are identified.

11. Consider the displacement from A_2 to A_1'. If we put $O_-O_+=c$, and take this directed line as initial line, A_2A_1' is represented by the vector $c\sqrt{3}\,.\exp(i\pi/6)$. This is the displacement for each point of $A_2B_2C_2$ when,

after successive rotations $+2\pi/3$, $-2\pi/3$ about O_-, O_+, it takes up the position $A_1'B_1'C_1'$. We may also note that $c\sqrt{3}$ is the length of OP.

If the configuration is extended by treating each of the new triangles as ABC has been treated, a pattern will result in which two other triangles (say, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$) have the same circumcircle as ABC, but with sides parallel to those of $A_1B_1C_1$, $A_2B_2C_2$. This leads to a new and simple construction for these triangles :

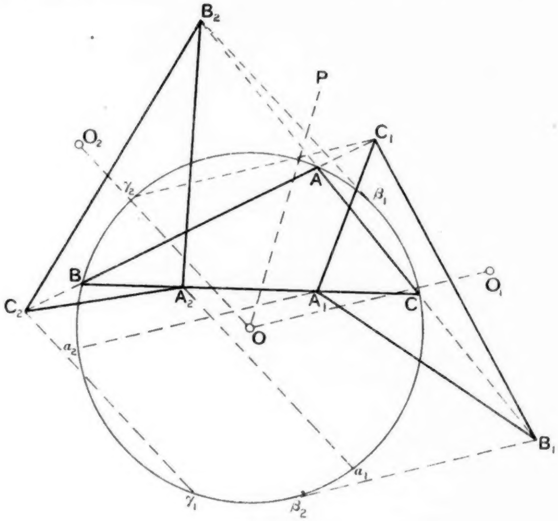

FIG. 4.

Starting with ABC, its circumcircle with centre O and its orthocentre P, mark off (Fig. 4) round the circle arcs $A\alpha_1$, $A\alpha_2$, $B\beta_1$, $B\beta_2$, $C\gamma_1$, $C\gamma_2$, each subtending $2\pi/3$ at O. With radius OP and centres α_2, β_2, γ_2 cut BC, CA, AB in A_1, B_1, C_1 ; with the same radius and centres α_1, β_1, γ_1 cut the same lines in A_2, B_2, C_2. These are the vertices of the two triangles already known ; and it is clear that the two sets of equal displacements α_1A_2, β_1B_2, γ_1C_2, OO_2 ; α_2A_1, β_2B_1, γ_2C_1, OO_1, make angles $\pm\pi/3$ with OP.

12. (β) *Cases* II, III.

In *Case* II, H coincides with the Brocard point Ω and K with Ω' (Fig. 5). We recall the known properties of these points :

$$\angle\Omega BC = \angle\Omega CA = \angle\Omega AB = \omega,$$
$$\angle\Omega' CB = \angle\Omega' AC = \angle\Omega' BA = \omega,$$
$$\cot\omega = \cot A + \cot B + \cot C.$$

The angle α for congruence is $\frac{1}{2}\pi - \omega$. The value $+(\frac{1}{2}\pi - \omega)$ yields a triangle $A_3B_3C_3$ (Fig. 5) ; the value $-(\frac{1}{2}\pi - \omega)$ yields a triangle coincident with ABC. It is evident that Ω is also the first Brocard point of XYZ for this case, and therefore also of $A_3B_3C_3$. Hence $A_3B_3C_3$ comes from ABC by rotation *about Ω positively through* $\pi - 2\omega$. There is, in general, no cyclic relation as in *Case* I (§ 9).

In *Case* III, H coincides with Ω' and K with Ω. Rotation of ABC about Ω' through $-(\pi-2\omega)$ yields the triangle $A_4B_4C_4$, while the second triangle of this case coincides with ABC.

Since $O\Omega=O\Omega'$, and $\angle\Omega O\Omega'=2\omega$, the rotations of ABC through $\pi-2\omega$ about Ω and through $-(\pi-2\omega)$ about Ω' will each bring the circumcentre O to the same point, say O_3, where $O\Omega O_3\Omega'$ is a rhombus ; and *the circumcentres of* $A_3B_3C_3$, $A_4B_4C_4$ coincide in O_3.

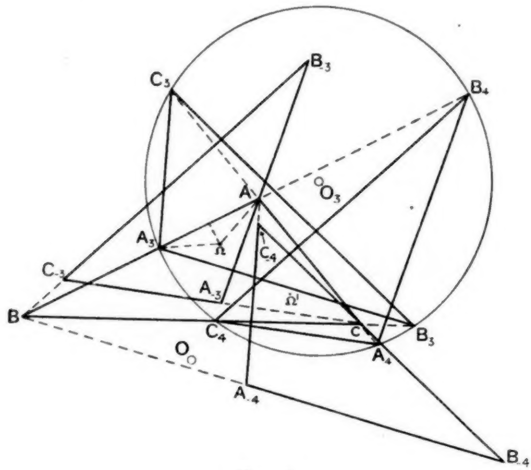

Fig. 5.

13. The following points may be noted :

(i) It is known that

$$\Omega A : (b/a) = \Omega B : (c/b) = \Omega C : (a/c)$$
$$= \Omega' A : (c/a) = \Omega' B : (a/b) = \Omega' C : (b/c),$$

hence
$$AA_3/AA_4 = A\Omega/A\Omega' = b/c ;$$

thus the triangle AA_4A_3 is similar to ABC ; and the same holds for BB_4B_3, CC_4C_3.

(ii) Of the triangles with vertices at the points $A, B, C, A_3, B_3, C_3, A_4, B_4, C_4$, six are mutually similar, three of them being congruent, and two of these lying on the same circumcircle.

(iii) A_3A_4, B_3B_4, C_3C_4 are antiparallels to BC, CA, AB in ABC.

14. By construction (B) we obtain $A_{-3}B_{-3}C_{-3}$, $A_{-4}B_{-4}C_{-4}$ with $B_{-3}C_{-3}$, $C_{-3}A_{-3}$, A_{-3}, B_{-3} passing respectively through B, C, A ; and $B_{-4}C_{-4}$, $C_{-4}A_{-4}$, $A_{-4}B_{-4}$ passing respectively through C, A, B. These circumscribed triangles may also be obtained, when $A_3B_3C_3$, $A_4B_4C_4$ are known, by drawing parallels to the sides of these latter through A, B, C. The five triangles Δ, Δ_3, Δ_4, Δ_{-3}, Δ_4, constitute a *five-triangle configuration* (§ 8).

15. (γ) *Cases* IV, V, VI.

In *Case* IV we have, reverting to Fig. 1, angles X, Y, Z respectively equal

to A, C, B ; hence

$$BH \sin B = y = 2R_1 \sin C, \quad CH \sin C = z = 2R_1 \sin B,$$

so that

$$BH = 2R_1 c/b, \quad CH = 2R_1 b/c\,;$$

then

$$a^2 = BC^2 = BH^2 + HC^2 - 2BH \,.\, HC \cos 2A$$
$$= 4R_1{}^2(b^4 + c^4 - 2b^2c^2 \cos 2A)/b^2c^2.$$

Thus

$$a^2b^2c^2 = 4R_1{}^2\{(b^2+c^2)^2 - (2bc \cos A)^2\}$$
$$= 4R_1{}^2(b^2 + c^2 + 2bc \cos A)(b^2 + c^2 - 2bc \cos A)$$
$$= 16R_1{}^2 m_a{}^2 a^2,$$

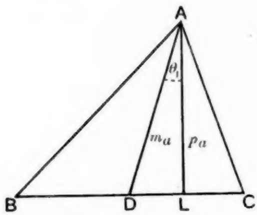

FIG. 6.

where (Fig. 6) m_a is the length of the median AD of ABC. But $bc = 2Rp_a$, where p_a is the altitude AL of ABC from A to BC. Hence $2Rp_a = 4R_1 m_a$,

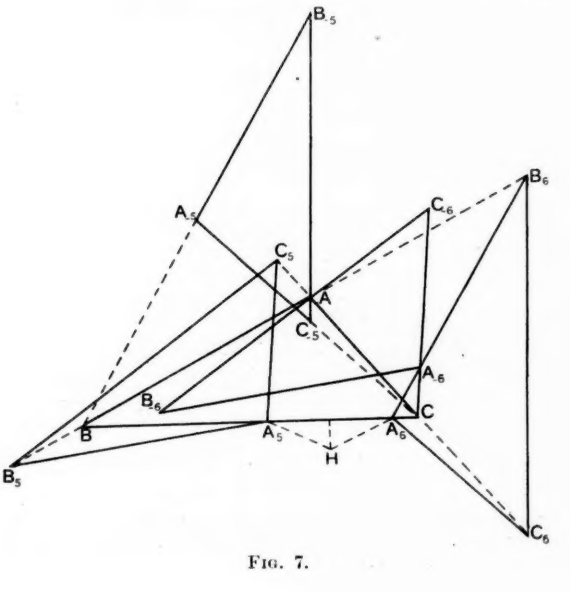

FIG. 7.

and the ratio of linear dimensions of the minimum triangle to the original is

$$R_1/R = p_a/2m_a = \tfrac{1}{2}\cos\theta_1,$$

where θ_1 is the angle DAL, between median and altitude at A. Therefore, to obtain triangles congruent to ABC, we shall draw HA_5, HA_6 making angles $\pm\sec^{-1}(2\sec\theta_1)$ with HX, and so for the other vertices (Fig. 7). The associated triangles $A_5'B_5'C_5'$, $A_6'B_6'C_6'$, may either be obtained by construction (B), or simply by drawing parallels to the sides of $\triangle_5$, $\triangle_6$, through the vertices of $\triangle$.

Cases V, VI follow the lines of IV, *mutatis mutandis*. There do not seem to be any striking relations between the pairs $\triangle_5$, $\triangle_6$; $\triangle_7$, $\triangle_8$; $\triangle_9$, $\triangle_{10}$; and they are drawn in separate diagrams (Figs. 7, 8, 9).

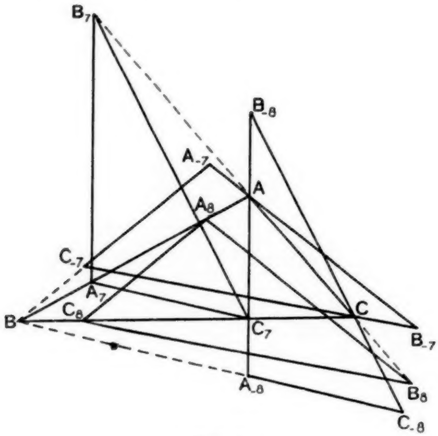

FIG. 8.

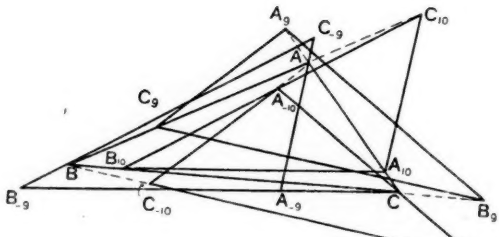

FIG. 9.

The *five-triangle configuration* appears in these as in (α), (β) above; but with this difference, that in IV, V, VI the derived triangles are in the opposite sense of rotation from the original. D. G. T.

A CHAIN RULE FOR USE WITH DETERMINANTS AND PERMUTATIONS.

By N. Chater and W. J. Chater.

To introduce the proposed rule we make use of it to settle the signs of terms in the expansion of determinants such as :

$$A = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix}$$

where the symbol a_s^r for the element has superscript r denoting rows and subscript s denoting columns.

" *Chains* " *and* " *Links* ".

A term of the expansion of A such as $a_2^1 a_1^2 a_3^3$ is divided into parts called " chains ", *i.e.* into $a_2^1 a_1^2$ and a_3^3. The first of these parts is a chain of two " links ", while the second may be called a one-linked chain.

In a chain such as $a_2^1 a_1^2$, the superscript of the factor or link at the beginning equals the subscript of the link at the end, whilst the subscript of a preceding link is the same as the superscript of the one immediately following it. An example of a three-linked chain is the term $a_2^1 a_3^2 a_1^3$. The superscript we start with is 1 and equals the subscript we end on, whilst the linkage is from subscript 2 to superscript 2 between the first two factors and from subscript 3 to superscript 3 between the second two factors.

To separate the chains from one another in any term of the expansion we employ the multiplication sign : ($\times$).

Thus the term $a_2^1 a_1^2 a_3^3$, when separated into chains appears as $a_2^1 a_1^2 \times a_3^3$.

The special form of the chain rule for the third-order determinant (A) is :

In a determinant of order *three*, terms in the expansion which consist of 3 chains or 1 chain have the positive sign, whilst terms consisting of 2 chains have the negative sign.

The sign of $a_2^1 a_1^2 a_3^3$ $(= a_2^1 a_1^2 \times a_3^3)$ is the sign to be given to a term of two chains, *i.e.* the negative sign, and we write $-a_2^1 a_1^2 a_3^3$.

The term $a_1^1 a_2^2 a_3^3$, the product of the three elements of A's principal diagonal, is separated into $a_1^1 \times a_2 \times a_3^3$.

It consists of 3 chains each of a single link, whence by the rule above its sign is positive.

In a determinant of order *four*, terms in the expansion having 4 or 2 chains are positive, whilst those having 3 chains or 1 chain are negative.

The general chain rule is as follows :

In a determinant of odd order, terms in the expansion consisting of any odd number of chains have the positive sign, whilst those with any even number of chains have the negative sign. Alternatively, in a determinant of even order, terms of the expansion containing any even number of chains have the positive sign, whilst those containing an odd number of chains have the negative sign.

The above chain rule gives results which correspond completely with those obtained from transposition rules or from diagrams of intersecting lines.

Note : The " sign " referred to above is independent, of course, of that borne by the element itself.

In determinants like B below, the symbols of elements may be converted into those of the style of A above by introducing appropriate superscripts :

a^1 for a, a^2 for b, a^3 for c. A term in B's expansion such as $c_1a_2b_3$ may be converted into $a^3_1a^1_2a^2_3$ prior to consideration of its sign by the chain rule.

$$B = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Further examples.

Take $a^1_1a^2_2a^3_4a^4_3$ from the expansion of the fourth-order determinant. Separated as chains it becomes : $a^1_1 \times a^2_2 \times a^3_4a^4_3$—*i.e.* an odd number of chains in a term of the expansion of a determinant of even order. By the rule above, its sign is negative, and we write $-a^1_1a^2_2a^3_4a^4_3$.

If we take the term : $a^1_4a^2_3a^3_5a^4_2a^5_1$ from the expansion of a fifth-order determinant, we have on re-arranging the factors to form chains : $a^1_4a^4_2a^2_3a^3_5a^5_1$, viz. a single chain ; 5-linked. The odd number of chains in a term in the expansion of a determinant of odd order requires a positive sign, so we write : $+a^1_4a^2_3a^3_5a^4_2a^5_1$.

The chain rule places the leading term in the expansion of a determinant, *i.e.* terms such as $a^1_1a^2_2a^3_3$ or $a^1_1a^2_2a^3_3a^4_4$, etc., in a place of special importance, as suggesting the sign for other terms of the expansion.

Thus $a^1_1 \times a^2_2 \times a^3_3$ (odd chains) suggests that odd chains in other terms of the expansion carry the same sign as the leader, viz. the positive sign. In the expression $a^1_1a^2_2a^3_3a^4_4$ $(= a^1_1 \times a^2_2 \times a^3_3 \times a^4_4)$ there is an even number of chains, suggesting that all other terms of the expansion with even chains carry the leading term's positive sign. In any particular order of determinant the leading term—product of elements down the principal diagonal—bears the positive sign, and its number of chains odd or even supplies the sign rule for any other term of the expansion.

In the special examples below, a term derived from the leading one of a determinant of order four by an odd number of transpositions produces an odd number of chains, whilst a term produced from the leading term by an even number of transpositions produces an even number of chains. Leading term $= a^1_1a^2_2a^3_3a^4_4$.

No. of transpositions by single steps	Term	Term in chains	No. of chains
1 (odd)	$a^1_1a^2_2a^4_3a^3_4$	$a^1_1 \times a^2_2 \times a^4_3a^3_4$	3 (odd)
2 (even)	$a^1_1a^4_2a^2_3a^3_4$	$a^1_1 \times a^4_2a^2_3a^3_4$	2 (even)
3 (odd)	$a^4_1a^1_2a^2_3a^3_4$	$a^4_1a^1_2a^2_3a^3_4$	1 (odd)
4 (even)	$a^4_1a^2_2a^1_3a^3_4$	$a^4_1a^1_3a^3_4 \times a^2_2$	2 (even)
5 (odd)	$a^4_1a^2_2a^3_3a^1_4$	$a^4_1a^1_4 \times a^2_2 \times a^3_3$	3 (odd)
6 (even)	$a^2_1a^4_2a^3_3a^1_4$	$a^2_1a^1_4a^4_2 \times a^3_3$	2 (even)

Thus odd in chains follows odd in transpositions, while even in transpositions is followed by even in chains.

With a fifth-order term such as : $a^1_1a^2_2a^3_3a^4_4a^5_5$, if we make an even number of transpositions the number of chains is odd. Thus two transpositions gives $a^1_1a^2_2a^5_3a^3_4a^4_5 = a^1_1 \times a^2_2 \times a^5_3a^3_4a^4_5$. Odd chains in a term in the expansion of a determinant of odd order gives a positive sign corresponding to the sign for even transpositions. An odd number of transpositions as in $a^2_1a^5_2a^1_3a^4_4a^3_5$ (five transpositions) gives in chains $a^2_1a^1_3a^3_5a^5_2 \times a^4_4$ (2 chains).

Five transpositions gives a negative sign ; so does even chain number in an odd-order expansion.

We write out the expansion of :

$$\begin{vmatrix} a^1_1 & a^1_2 & a^1_3 & a^1_4 \\ a^2_1 & a^2_2 & a^2_3 & a^2_4 \\ a^3_1 & a^3_2 & a^3_3 & a^3_4 \\ a^4_1 & a^4_2 & a^4_3 & a^4_4 \end{vmatrix}$$

using the appropriate form of the chain rule to settle the signs. The product of the elements down the principal diagonal is $a^1_1a^2_2a^3_3a^4_4$ or $a^1_1 \times a^2_2 \times a^3_3 \times a^4_4$ (4 chains). By the rule with this order of determinant, terms with 4 chains or 2 chains are positive. We use the leading term above to obtain all the other terms of the expansion in the following way : Keep the superscripts always 1, 2, 3, 4 in the order they are in the leading term. Fix the subscript 1 and permute the subscripts 2, 3, 4.

With these fixtures and permutations we obtain :

	Term	Expressed in chains	No. of chains	Sign
Leading term - - -	$a^1_1a^2_2a^3_3a^4_4$	$a^1_1 \times a^2_2 \times a^3_3 \times a^4_4$	4	+
Here,	$a^1_1a^2_2a^3_4a^4_3$	$a^1_1 \times a^2_2 \times a^3_4a^4_3$	3	−
superscripts unchanged	$a^1_1a^2_3a^3_2a^4_4$	$a^1_1 \times a^2_3a^3_2 \times a^4_4$	3	−
subscript 1 fixed	$a^1_1a^2_3a^3_4a^4_2$	$a^1_1 \times a^2_3a^3_4a^4_2$	2	+
subscripts 2, 3, 4 per-	$a^1_1a^2_4a^3_2a^4_3$	$a^1_1 \times a^2_4a^4_3a^3_2$	2	+
muted	$a^1_1a^2_4a^3_3a^4_2$	$a^1_1 \times a^2_4a^4_2 \times a^3_3$	3	−

Thus this set of six terms is :

$$a^1_1a^2_2a^3_3a^4_4 - a^1_1a^2_2a^3_4a^4_3 - a^1_1a^2_3a^3_2a^4_4 + a^1_1a^2_3a^3_4a^4_2 + a^1_1a^2_4a^3_2a^4_3 - a^1_1a^2_4a^3_3a^4_2.$$

A further set of six terms in the expansion is obtainable from the previous set by keeping the superscripts 1, 2, 3, 4 as before and stepping-up the subscripts 1, so that the subscripts 1, 2, 3, 4 become 2, 3, 4, 1. This gives us a set of six terms, two of which are : $a^1_2a^2_3a^3_4a^4_1$ and $a^1_2a^2_3a^3_1a^4_4$.

Written in chains these become : $a^1_2a^2_3a^3_4a^4_1$ (1 chain, whence negative),
$a^1_2a^2_3a^3_1 \times a^4_4$ (2 chains, whence positive).

A further six terms are obtainable from this second set by keeping the superscripts unchanged and stepping-up the subscripts 1. Thus the terms $a^1_2a^2_3a^3_4a^4_1$ and $a^1_2a^2_3a^3_1a^4_4$ become $a^1_3a^2_4a^3_1a^4_2$ and $a^1_3a^2_4a^3_2a^4_1$.
The latter written as chains are : $a^1_3a^3_1 \times a^2_4a^4_2$ (2 chains, whence positive),
$a^1_3a^3_2a^2_4a^4_1$ (1 chain, whence negative).

The last set of six is derived similarly. The two terms of it corresponding to the two immediately above are $a^1_4a^2_1a^3_2a^4_3$ and $a^1_4a^2_1a^3_3a^4_2$, which are
$a^1_4a^4_3a^3_2a^2_1$ (1 chain, whence negative),
and $a^1_4a^4_2a^2_1 \times a^3_3$ (2 chains, whence positive).

If we take the above terms together for comparison we have :

1st pair - - - $a^1_1a^2_2a^3_3a^4_4$, positive
$a^1_1a^2_2a^3_4a^4_3$, negative
2nd pair - - $a^1_2a^2_3a^3_4a^4_1$, negative
(1st stepping-up) $a^1_2a^2_3a^3_1a^4_4$, positive
3rd pair - - - $a^1_3a^2_4a^3_1a^4_2$, positive
(2nd stepping-up) $a^1_3a^2_4a^3_2a^4_1$, negative

4th pair - - - $a^1_4 a^2_1 a^3_2 a^4_3$, negative
(3rd stepping-up) $a^1_4 a^2_1 a^3_3 a^4_2$, positive

Whence of the 24 terms in the expansion, 12 are positive and 12 negative, and the sign settled by the chain rule for the first set of 6 terms passes to and fro from positive to negative or *vice versa* for corresponding terms of successive sets as the stepping-up process proceeds.

We use the chain rule also to settle the sign of a determinant derived from an initial one by interchange of rows or columns.

Suppose that in a determinant of order 3, the first and third columns are interchanged and then the second and third rows.

The leading term $a^1_1 a^2_2 a^3_3$ of the original determinant becomes, first, $a^1_3 a^2_2 a^3_1$, and then passes to $a^1_3 a^3_2 a^2_1$:

$$\begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix} \to \begin{vmatrix} a^1_3 & a^1_2 & a^1_1 \\ a^2_3 & a^2_2 & a^2_1 \\ a^3_3 & a^3_2 & a^3_1 \end{vmatrix} \to \begin{vmatrix} a^1_3 & a^1_2 & a^1_1 \\ a^3_3 & a^3_2 & a^3_1 \\ a^2_3 & a^2_2 & a^2_1 \end{vmatrix}$$

Leading terms: $a^1_1 a^2_2 a^3_3 \to a^1_3 a^2_2 a^3_1 \to a^1_3 a^3_2 a^2_1$.

Now $a^1_3 a^3_2 a^2_1$ is a one-chain expression, and in a determinant of order 3 decides the sign as positive. With this sign the final determinant has the same value as the initial one.

If rows one and two are first interchanged, $a^1_1 a^2_2 a^3_3$ becomes $a^2_1 a^1_2 a^3_3$, and if the third column is now brought into the first position we have $a^3_1 a^2_2 a^1_3$. In chains this is: $a^3_1 a^1_3 \times a^2_2$, whence the sign is negative by the rule. The derangements in question are shown below:

$$\begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix} \to \begin{vmatrix} a^2_1 & a^2_2 & a^2_3 \\ a^1_1 & a^1_2 & a^1_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix} \to \begin{vmatrix} a^2_3 & a^2_1 & a^2_2 \\ a^1_3 & a^1_1 & a^1_2 \\ a^3_3 & a^3_1 & a^3_2 \end{vmatrix}$$

The leading term of the last determinant is: $a^2_3 a^1_1 a^3_2 = a^1_1 \times a^2_3 a^3_2$, as chains. Whence its sign is negative in the original determinant. Thus the final determinant $= -$ (the original one).

The chain rule applied in the way shown allows us to determine the sign changes of a determinant due to derangements of rows and columns.

Permutations.

We may use the chain rule in the discussion of permutations. To make use of the rule we employ the notation $\dfrac{43215}{12345}$ for a permutation, where the numbers in the denominator denote places and those in the numerator the values occupying the place they stand above. Thus the value 4 is in the first place, the value 3 in the second place, and so on. Ordinarily the above permutation would appear as 43215.

Permutations belong to the class C_+ or C_-. The class is easily settled by the chain rule.

Thus the permutation $\dfrac{43215}{12345}$ (of odd order) is first split into chains. We begin with the 4 in the numerator which has 1 in the denominator immediately below it. We next seek a 1 in the numerator to assure the linkage. Below this 1, chosen from the numerator, there stands a 4 in the denominator. Thus we have $\dfrac{41}{14}$ as our permutation chain, beginning with a numerator

quantity 4, the same as the final denominator one, and with linkage quantity 1 joining the two parts of the chain.

In this way the above permutation may be written $\frac{41}{14} \times \frac{32}{23} \times \frac{5}{5}$, *i.e.* a three-chain expression in a fifth-order permutation, *i.e.* an *odd* chain expression in an *odd*-order permutation. This makes the class C_+.

The general classification rule is as follows:

Odd-order permutations consisting of an odd number of chains are C_+, those consisting of an even number of chains are C_-. Alternatively, even-order permutations consisting of an even number of chains are C_+, whilst those consisting of an odd number of chains are C_-.

In the permutation $\frac{43215}{12345}$ the " values " in the numerator simulate subscripts, while the figures of the denominator, representing the " places " occupied by the " values ", correspond to subscripts.

There is no need to begin with the 4 of the numerator in separating the permutation into chains.

We might have started with the 2, say, and so obtained $\frac{23}{32} \times \frac{5}{5} \times \frac{41}{14}$—the same number of chains as before, leading to the same class.

The permutation $\frac{53421}{12345}$ gives $\frac{51}{15} \times \frac{324}{243}$, *i.e.* 2 chains, whence in a fifth-order permutation its class is C_-.

To compare the classes of 2 permutations $\frac{43215}{12345}$ and $\frac{53421}{12345}$ we write

$$\frac{43215}{12345} \Big/ \frac{53421}{12345} = \frac{43215}{53421},$$

which is equal to $\frac{4512}{5124} \times \frac{3}{3}$. The number of chains is 2. With fifth-order permutations the value 2 for the chains indicates a negative relation, *i.e.* that the permutations belong to different classes.

Again, consider $\frac{21534}{12345}$ and $\frac{12435}{12345}$.

Looking at these separately we have:

$$\frac{21}{12} \times \frac{534}{345} \text{ (2 chains, whence } C_-\text{)}$$

and

$$\frac{1}{1} \times \frac{2}{2} \times \frac{43}{34} \times \frac{5}{5} \text{ (4 chains, whence } C_-\text{).}$$

The two permutations are both in the C_- class.

Treating them relatively we have:

$$\frac{21534}{12345} \cdot \frac{12345}{12435} = \frac{21534}{12435} = \frac{21}{12} \times \frac{54}{45} \times \frac{3}{3}.$$

There are three chains in a five-order scheme, so that a positive relationship holds and the two permutations belong to the same class.

Permutation conjugates and self-conjugates.

A permutation's conjugate is defined as one in which the value is changed into place and the place into value.

Thus if $P = \frac{43215}{12345}$, P's conjugate is $\frac{12345}{43215} = \frac{43215}{12345}$, where we have arranged

the places in the denominator in the standard order 1, 2, 3, 4, 5, and written the values filling the places in the numerator. In the case above, the conjugate of P is the same as the original permutation. Such permutations are described as self-conjugate ones.

The permutation $P=\frac{21534}{12345}$ has as its conjugate $\frac{12345}{21534}=\frac{21453}{12345}$, which differs from the one from which it was derived, whence $\frac{21534}{12345}$ is not a self-conjugate permutation.

If we write the self-conjugate permutation $\frac{43215}{12345}$ in chains we obtain $\frac{41}{14}\times\frac{32}{23}\times\frac{5}{5}$. No chain has more than two links. The permutation $\frac{21534}{12345}$ may be written $\frac{21}{12}\times\frac{534}{345}$. One of its chains has three links. It is not self-conjugate.

The connection between self-conjugacy and links may be expressed as follows: Permutations containing chains of more than two links are not self-conjugate. The relative principle may be called in here also.

If we know that $\frac{43215}{12345}$ is self-conjugate, and want to know whether $\frac{21534}{12345}$ is self-conjugate or not, we consider:

$$\frac{43215}{12345}\cdot\frac{12345}{21534}=\frac{43215}{21534}.$$

In chains: $\frac{425}{254}\times\frac{31}{13}$. There is a chain of more links than two. The two permutations differ in their relation to their conjugates, so that if the first permutation is known to be self-conjugate, the second one is not self-conjugate.

We construct two self-conjugate permutations:

$$\frac{1}{1}\times\frac{2}{2}\times\frac{3}{3}\times\frac{54}{45}\quad\text{and}\quad\frac{1}{1}\times\frac{32}{23}\times\frac{54}{45},$$

i.e. $\frac{12354}{12345}$ and $\frac{13254}{12345}$. Neither of these contains chains of more than two links. They are both self-conjugate permutations.

The relative treatment gives

$$\frac{12354}{12345}\cdot\frac{12345}{13254}=\frac{12354}{13254}=\frac{1}{1}\times\frac{23}{32}\times\frac{5}{5}\times\frac{4}{4}.$$

In this last expression there is no chain larger than a two-linked one, whence the relations to their conjugates of the two permutations is the same. If the first is self-conjugate, as it is, the relative treatment shows that the second also is self-conjugate.

To obtain the number of self-conjugate permutations, we discuss first the permutation of order 5.

The number of self-conjugate permutations with five one-linked chains is $1={}^5C_0$.

The number with 1 two-linked chain and 3 one-linked ones is ${}^5C_2=\frac{2!}{2!\,1!}\,{}^5C_2$.

The number of permutations with 2 two-linked chains and 1 one-linked one is the number of ways of choosing four from five multiplied by three. If $a_2^1a_1^2\times a_4^3a_3^4\times a_5^5$ represents one choice with a_5^5 as the one-linked chain, we can obtain $a_3^1a_1^3\times a_4^2a_2^4\times a_5^5$ and $a_4^1a_1^4\times a_2^3a_3^2\times a_5^5$, *i.e.* three different selections with the single-linked chain a_5^5, whence the total selections will be ${}^5C_1\times 3=3\,.\,{}^5C_4$.

We shall replace 3 by $4!/2^2 \,.\, 2!$ to which it is equal, and obtain for the total number of self-conjugate permutations of order five the value

$$\begin{aligned} &{}^5C_0 + \frac{2!}{2^1 \,.\, 1!}\,{}^5C_2 + \frac{4!}{2^2 \,.\, 2!}\,{}^5C_4 \\ &= 1 + 10 + 15 = 26. \end{aligned}$$

Similarly the total number of self-conjugate permutations of orders 6 and 7 are respectively :

$${}^6C_0 + \frac{2!}{2^1 \,.\, 1!}\,{}^6C_2 + \frac{4!}{2^2 \,.\, 2!}\,{}^6C_4 + \frac{6!}{2^3 \,.\, 3!}\,{}^6C_6$$

and

$${}^7C_0 + \frac{2!}{2^1 \,.\, 1!}\,{}^7C_2 + \frac{4!}{2^2 \,.\, 2!}\,{}^7C_4 + \frac{6!}{2^3 \,.\, 3!}\,{}^7C_6,$$

where here $5 \,.\, 3$ is replaced by $6!/2^3 \,.\, 3!$ to which it is equal.

The first of these two last expressions has the value

$$1 + 1 \,.\, 15 + 3 \,.\, 15 + 15 \,.\, 1 = 76,$$

and the second has the value

$$1 + 1 \,.\, 21 + 3 \,.\, 35 + 15 \,.\, 7 = 232.$$

Writing u_n for the number of self-conjugate permutations of order n, we have

$$u_7 = 232, \quad u_6 = 76, \quad u_5 = 26\,;$$

here

$$u_6 + (7-1)u_5 = 76 + 6 \,.\, 26 = 232 = u_7.$$

In general, the number u_n of self-conjugate permutations of order n is

$${}^nC_0 + \frac{2!}{2^1 \,.\, 1!}\,{}^nC_2 + \frac{4!}{2^2 \,.\, 2!}\,{}^nC_4 + \frac{6!}{2^3 \,.\, 3!}\,{}^nC_6 + \ldots + \frac{(2r)!}{2^r \,.\, r!}\,{}^nC_{2r} + \ldots,$$

or

$${}^nP_0 + \frac{1}{2^1 \,.\, 1!}\,{}^nP_2 + \frac{1}{2^2 \,.\, 2!}\,{}^nP_4 + \frac{1}{2^3 \,.\, 3!}\,{}^nP_6 + \ldots + \frac{1}{2^r \,.\, r!}\,{}^nP_{2r} + \ldots.$$

This series gives values which satisfy the relation

$$u_n = u_{n-1} + (n-1)u_{n-2}.$$

See the Rothe-Aitken problem (Turnbull, *The Theory of Determinants, Matrices and Invariants*, 1929, p. 29). The fact may be demonstrated as follows : let n be an odd integer. Then

$$\begin{aligned} u_n &= {}^nC_0 + \frac{2!}{2^1 \,.\, 1!}\,{}^nC_2 + \frac{4!}{2^2 \,.\, 2!}\,{}^nC_4 + \frac{6!}{2^3 \,.\, 3!}\,{}^nC_6 + \ldots \\ &= {}^nC_0 + 1 \,.\, {}^nC_2 + 1 \,.\, 3\,{}^nC_4 + 1 \,.\, 3 \,.\, 5 \,.\, {}^nC_6 + \ldots + 1 \,.\, 3 \,.\, 5 \ldots (n-2)\,{}^nC_{n-1}. \end{aligned}$$

Replacing n by $n+1$, $n-1$ respectively, we have

$$\begin{aligned} u_{n+1} = {}^{n+1}C_0 + 1 \,.\, {}^{n+1}C_2 + 1 \,.\, 3\,{}^{n+1}C_4 + \ldots + 1 \,.\, 3 \,.\, 5 \,.\, \ldots (n-2)\,{}^{n+1}C_{n-1} \\ + 1 \,.\, 3 \,.\, 5 \ldots n \,.\, {}^{n+1}C_{n+1}, \end{aligned}$$

$$u_{n-1} = {}^{n-1}C_0 + 1\,{}^{n-1}C_2 + 1 \,.\, 3\,{}^{n-1}C_4 + \ldots + 1 \,.\, 3 \,.\, 5 \ldots (n-2) \,.\, {}^{n-1}C_{n-1}.$$

If $u_{n+1} = u_n + n \,.\, u_{n-1}$, we must have :

$$\begin{aligned} &({}^{n+1}C_0 - {}^nC_0) + (1 \,.\, {}^{n+1}C_2 - 1 \,.\, {}^nC_2 - n\,{}^{n-1}C_0) \\ &\quad + (1 \,.\, 3\,{}^{n+1}C_4 - 1 \,.\, 3\,{}^nC_4 - n \,.\, 1\,{}^{n-1}C_2) \\ &\quad + (1 \,.\, 3 \,.\, 5\,{}^{n+1}C_6 - 1 \,.\, 3 \,.\, 5\,{}^nC_6 - n \,.\, 1 \,.\, 3\,{}^{n-1}C_4) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

$$+(1\,.\,3\,.\,5\ldots(n-2)\,{}^{n+1}C_{n-1}-1\,.\,3\,.\,5\ldots(n-2)\,{}^{n}C_{n-1}$$
$$-n\,.\,1\,.\,3\,.\,5\ldots(n-4)\,{}^{n-1}C_{n-3})$$
$$+(1\,.\,3\,.\,5\ldots n\,{}^{n+1}C_{n+1}-n\,.\,1\,.\,3\,.\,5\ldots(n-2)\,{}^{n-1}C_{n-1})$$
$$=0.$$

Using the well-known relation between combinations

$$^{n+1}C_x = {}^nC_x + {}^nC_{x-1},$$

the above becomes :

$$(1-1)+({}^nC_2+{}^nC_1-{}^nC_2-{}^nC_1)$$
$$+(1\,.\,3\,{}^nC_4+1\,.\,3\,{}^nC_3-1\,.\,3\,{}^nC_4-1\,.\,3\,{}^nC_3)$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+(1\,.\,3\,.\,5\ldots(n-2)n\,{}^{n+1}C_{n+1}-1\,.\,3\,.\,5\ldots(n-2)n\,{}^{n-1}C_{n-1})$$
$$=0.$$

With each of the bracketed quantities equal to zero, the relationship required is satisfied. Similarly for n an even integer.

Reverting to the expansion of a determinant of order 4, we have

$$a_1^1A_1^1+a_2^1A_1^2+a_3^1A_1^3+a_4^1A_1^4=a,$$

the value of the determinant. But as we saw previously, using certain fixtures and permutations six terms of the expansion are :

$$a_1^1(a_2^2a_3^3a_4^4-a_2^2a_4^3a_3^4-a_3^2a_2^3a_4^4+a_3^2a_4^3a_2^4+a_4^2a_2^3a_3^4-a_4^2a_3^3a_2^4).$$

Writing, as we have done above, A_i^j as the cofactor of a_j^i, we see that the cofactor of a_1^1, viz. A_1^1, is the quantity in the bracket, and that when the bracket is removed the six terms produced are $a_1^1A_1^1$.

Stepping-up the subscript of the first factor, we reach $A_2^1a_1^2$, where A_1^2 is the quantity in the bracket whose terms have superscripts the same as before, and subscripts stepped-up one, *i.e.*

$$A_1^2=(-a_3^2a_4^3a_1^4+a_3^2a_1^3a_4^4+a_4^2a_3^3a_1^4-a_4^2a_1^3a_3^4-a_1^2a_3^3a_4^4+a_1^2a_4^3a_3^4).$$

Similarly, A_1^3 and A_1^4 are obtained. The signs of A_1^3 follow those of A_1^1, those of A_1^4 follow those of A_1^2.

In the expansion of any $2n$-order determinant the terms in the expansion of the co-factors A_r^s follow each other in sign if $r+s$ is even, while those in A_ρ^σ follow each other in sign if $\rho+\sigma$ is odd, but differ from those in A_r^s.

If the cofactor of a_j^i is taken as A_j^i the stepping-up connection is directly connected with the alteration in the suffixes i, j instead of inversely as above.

In conclusion, we examine the signs of products in Laplace expansions of a determinant. Suppose we require the sign of such a product as

$$\begin{vmatrix} a_3^2 & a_5^2 \\ a_3^4 & a_5^4 \end{vmatrix} \quad \begin{vmatrix} a_1^1 & a_2^1 & a_4^1 \\ a_1^3 & a_2^3 & a_4^3 \\ a_1^5 & a_2^5 & a_4^5 \end{vmatrix}$$

in the Laplace expansion of a determinant of order 5 in terms of determinants of orders 2 and 3.

The product of terms in the principal diagonals of the two factors is

$$a_3^2a_5^4a_1^1a_2^3a_4^5.$$

Arranged in chains, this is $a_3^2a_2^3\times a_5^4a_4^5\times a_1^1$, *i.e.* 3 chains. As the determinant being expanded is of order 5 this term must be positive, so that the sign of the determinantal product above is accordingly likewise positive.

The sign of

$$\begin{vmatrix} a_3^2 & a_5^2 \\ a_3^5 & a_5^5 \end{vmatrix} \begin{vmatrix} a_1^1 & a_2^1 & a_4^1 \\ a_1^3 & a_2^3 & a_4^3 \\ a_1^4 & a_2^4 & a_4^4 \end{vmatrix}$$

depends on the sign to be attached to : $a_3^2 a_5^5 a_1^1 a_2^3 a_4^4$, the product of elements in the principal diagonals. Arranged as chains this is : $a_3^2 a_2^3 \times a_1^1 \times a_4^4 \times a_5^5$. This is a 4-chain term, and is therefore negative. Whence the sign of the determinantal product is likewise negative.

When $| a_1 b_2 c_3 d_4 e_5 f_6 |$ is expanded in various developments, what sign should be attached to each of the following?

$$| b_2 d_4 | \; | a_1 c_3 e_6 f_5 |, \; | a_6 f_5 d_4 | \; | b_3 c_2 e_1 |, \; | a_4 b_5 c_6 | \; | d_1 e_3 f_2 |.$$

Turnbull, *loc. cit.*, p. 25, Ex. 3.

Translated into a_s^r symbols we determine the signs of :

$$| a_2^2 a_4^4 | \; | a_1^1 a_3^3 a_6^5 a_5^6 |, \; | a_6^1 a_5^6 a_4^4 | \; | a_3^2 a_2^3 a_1^5 |, \; | a_4^1 a_5^2 a_6^3 | \; | a_1^4 a_3^5 a_2^6 |$$

which are representative of various developments of $| a_1^1 a_2^2 a_3^3 a_4^4 a_5^5 a_6^6 |$.

$$| a_2^2 a_4^4 | \; | a_1^1 a_3^3 a_6^5 a_5^6 |$$

gives as its product of principal diagonal terms $a_2^2 \times a_4^4 \times a_1^1 \times a_3^3 \times a_6^5 a_5^6$ (5 chains in a 6-order determinant), sign negative.

$$| a_6^1 a_5^6 a_4^4 | \; | a_3^2 a_2^3 a_1^5 |$$

gives us the corresponding product $a_6^1 a_5^6 a_1^5 \times a_4^4 \times a_3^2 a_2^3$ (3 chains in a 6-order determinant), sign negative.

$$| a_4^1 a_5^2 a_6^3 | \; | a_1^4 a_3^5 a_2^6 |$$

gives $a_4^1 a_1^4 \times a_5^2 a_3^5 a_6^3 a_2^6$ (2 chains in a 6-order determinant), sign positive.

These correspond exactly with the signs given by the author of the above textbook.

Summary. We have introduced a chain rule to settle the sign of terms in the expansion of determinants.

In a determinant of *odd* order any term of the expansion which contains an *odd* number of chains has the positive sign, and in a determinant of *even* order a term of the expansion which contains an *even* number of chains also receives the positive sign. Other terms are negative.

This rule has been used to settle the sign of determinants relative to some original determinant from which they were derived by derangement of rows and columns, and is applied in permutations directly and relatively to settle the class C_+ or C_- to which they belong. Conjugate and self-conjugate permutations are discussed, and the relation between the amount of linkage and self-conjugacy is noted.

Permutations written in the mode $\dfrac{abcde}{12345}$, etc., are self-conjugate if the chains into which they may be divided contain not more than two links.

The number of self-conjugate permutations satisfying

$$U_n = U_{n-1} + (n-1)\, U_{n-2} \quad \text{(Rothe, Aitken)}$$

may be calculated from :

$$^nC_0 + \frac{2!}{2^1 1!}\, ^nC_2 + \frac{4!}{2^2 2!}\, ^nC_4 + \frac{6!}{2^3 3!}\, ^nC_6 + \dots .$$

The chain rule is used to settle the sign of the determinantal expressions in various Laplace developments.

N. C. & W. J. C.

REVIEWS.

Fundamental Theory. By the late Sir A. S. Eddington. Edited by Sir Edmund Whittaker. Pp. viii, 292. 25s. 1946. (Cambridge University Press)

It is probably true in general that, if readers of a particular book feel considerable need for a review, then the difficulty in reviewing that book is also considerable. Now a reviewer is himself one of the readers (at any rate in certain cases, according to the Editor (Vol. 30, p. 193)). As a reader of this book by Eddington, I find that he apparently just sits down and calculates the values of all the fundamental constants of physics without any appeal to quantitative observations (except in order to express his results in customary units), and I ask, Is this the greatest book ever written? But then, when I try to understand how Eddington has done what he says he has, I meet language like this, " The bi-particle of multiplicity 136 carries an unspecialised element of the excitation energy tensor added to the zero-temperature uranoid ; and this is divided into two parts, one part being stabilised as a V_{10} particle (proton or electron) and the other as a V_1 hole ". I ask, Does this sort of thing mean anything at all? I read on and am confronted with all the paraphernalia of probability distributions, symbolic frames and so on, applied to " intracules " and " extracules " and other notions I have never heard of before. Instinctively, whether or not rationally, I ask, Are the very foundations of physics really so complicated and difficult? Do they really require what seems to be a new language for their explanation, or ought they not rather to be characterised by some grand simplicity? I am prepared to find that it is a bit tricky to determine, say, the mass-ratio of the proton and the electron by pure calculation and to get the right answer (1836.34), but I do feel an urge to ask, Ought there not to be some beautifully simple argument which makes it obvious that the proposed calculation is a sensible one to undertake? These are all very big questions and, still as a reader, I am inclined to think that never have I so much wanted the guidance of a review.

As an individual invited to write a review, I am even more definitely inclined to think that never can there have been a book which it would be more difficult to review. Had the request come from any journal other than the *Gazette*, I personally could not have undertaken the task. I can accept the present invitation only because, in the case of the *Gazette*, the main task has already been accomplished!

Sir Edmund Whittaker has given in his article on " Eddington's theory of the constants of nature " * what is surely the best summary written by anyone, not excepting Eddington himself, of the aims and achievements of the work. One would urge every reader of the book to begin by reading that article ; one regrets only that it has not been reprinted as an introduction in the book itself. It certainly renders entirely superfluous any feebler attempt at a general summary of Eddington's work in the pages of the *Gazette*. All that it leaves one to do is to offer a few comments resulting from the fact that the article was written before the book itself was published.

The manuscript of the book was largely complete at the time of Eddington's death in November 1944, and Sir Edmund Whittaker accepted an invitation to edit it for publication. Sir Edmund's modest editorial preface provides but a slight indication of the debt which the scientific world owes to him for what he has done.

The way in which the author proposed to complete the book is indicated

* *Mathematical Gazette*, 29 (1945), 137–44.

by a note "probably written on the last day of Eddington's working life" reproduced at the end of the book as he left it. There were to have been three further chapters, but the last was to be only a summary, and one of the others an expansion of a paper already published in *Proc. Cambridge Phil. Soc.*, and consequently reprinted as an Appendix to the present volume. The one other missing chapter was evidently going to be quite a short one on some further applications of the theory. So we doubtless have in the book almost everything that Eddington intended should be there. When his editor remarks that "the work is complete in itself" he is, however, referring rather to the fact that it "practically replaces all the author's previous writings on his theory of the constants of Nature". It is as though Eddington, convinced all along that he had seen the truth, was nevertheless not convinced that he had put it across, and so was here making a supreme effort to give afresh a coherent account of everything that he held to be fundamental in physics. The description of the work, which one repeatedly hears, as his "scientific testament" is almost inescapable. It may be imagination, or it may be merely because Eddington would have redrafted the presentation had he lived, but one does somehow get the impression from its style that he was writing under some sense of urgency. There is at times a terseness one was less aware of in his earlier works. Possibly Udny Yule's method of statistical study of vocabulary would determine whether there is anything in this impression.

The first five chapters of *Fundamental Theory* are a revision and extension of Eddington's Dublin Lectures of 1942 (published by the Dublin Institute for Advanced Studies, 1943). Partly in support of the view just expressed, one may quote a sentence from his introductory paragraph in these Lectures: "But in fourteen years [since 1928] I have never had the smallest doubt that the direction which I took in 1928 was the one which leads to the unified relativity and quantum theory". This unification is, of course, the object of Eddington's work. He pursued it, not for the sake of the formal tidiness in having one theory instead of two, but because he saw in it the only hope of understanding the fundamental *structure* of physics.

The theme of these chapters is what Eddington calls the *statistical extension* of relativity theory. Some extension of the notions of molar relativity theory is obviously needed in order to connect it with microscopic physics, *i.e.* quantum theory. The extension which Eddington here chooses is to develop a relativity theory which takes account of the principle of uncertainty in quantum theory. It is called statistical because that principle allows us to assign to a particle in general only a probability distribution of position and velocity. It leads very quickly to a host of definitions of new concepts which the extension appears to demand and which make it impossible briefly to summarise the technique of the theory. But it has to be said that this way of fusing relativity and quantum theory leads Eddington to conclude that certain features of the two theories are different descriptions of the same components in the fundamental structure of physics. This, of course, has to be the case if the programme is to succeed in yielding new relations amongst the natural constants from which the constants themselves are to be calculated. Those connections which Eddington claims to establish are at first sight surprising, but at the same time satisfying in the sense that their assertion both stimulates one's awareness of gaps to be filled in physical theory and also appeals to one as a natural way of filling the gaps. For instance, he asserts that "curvature in relativity and wave functions in quantum theory are alternative ways of representing distribution of energy and momentum", and this in due course leads to "exclusion is a wave mechanical substitute for gravitation". I do not know if it is entirely misleading to do so, but it is certainly very tempting to describe what Eddington does as the establishment of a complete set of "correspondence principles" between relativity and

quantum theory. Bohr's original Correspondence Principle might perhaps be said to provide the unification of classical molar theory and the old quantum theory; in a broadly analogous way, Eddington's correspondences may be said to give his unification of molar relativity theory and current quantum mechanics.

If this view is defensible, it tends to dispose of one's hankering after naïve simplicity in the foundations of physics. For, to refer again to the instance quoted, both "exclusion" and "gravitation" are difficult concepts even in their usual contexts, and one should expect any correspondence between them to involve yet more difficult concepts. It may be possible to manipulate the concepts more simply than Eddington does, but their intrinsic difficulty seems likely to persist.

The next three chapters give a condensed account of the contents of the author's *Relativity Theory of Protons and Electrons* (1936), of which I think a reasonably fair notice was given in the *Gazette* 21 (1937), 232–6. The theme is what Eddington has since called the *spin extension* of relativity theory to distinguish it from the statistical extension. Instead of selecting the uncertainty associated with a particle in quantum theory as the primary feature which relativity must be extended to incorporate, Eddington finds that it is possible alternatively to select its spin characteristics. The implication of his work is that if one feature is properly allowed for then the other is automatically included. He evidently regarded his statistical extension as giving more immediate insight and the spin extension as giving more immediately the appropriate calculus, that of his E-numbers, or as his editor calls it, *sedenion* analysis.

Of his account of the Dublin lectures, Eddington wrote: "The article concludes at the point where it would, I think, cease to be profitable to develop the statistical theory alone; that is to say, the natural continuation would be to give the full development of the spin theory, and then to pass on to problems in which both are applied concurrently". This is the plan he did in fact carry out in *Fundamental Theory*, and its remaining four chapters have the character implied by the last clause. It is worth quoting Eddington's explanation of this arrangement of the work. He thought it better to begin with his statistical theory because it supplies the definition of the new concepts, while the spin theory "is highly mathematical, and is likely to degenerate into pure algebra unless it is guided by a clear understanding of the concepts to which it is applied. I regard the introductory statistical part of the theory as the more difficult, because we have to use our brains all the time. Afterwards we can use mathematics instead".

For the immensely impressive results of the whole work, one must again refer to Whittaker's article. But for other reasons it seems desirable to refer for a moment to the Appendix already mentioned. It is on the evaluation of the cosmical number N, of which the significance is explained by Whittaker. It contains perhaps the clearest available account of some of Eddington's views on epistemology. But it is to a subsidiary feature to which I venture now to call attention. After explaining what he wants to show, he makes this startling confession: "A logically complete demonstration, if it is possible, would be extremely prolix, and it is not the sort of problem I could myself attempt. But I shall try to show that at each stage the investigation is being driven by its own momentum—that the moves leading to a universe of N particles are forced". If one is not entirely mistaken, this throws much light on Eddington's attitude towards his work. He evidently had the uttermost confidence in its general correctness, but at the same time wished to make no extravagant claims regarding the rigour of his detailed arguments. When, all too often, we come upon what seems to be a hiatus in the logical development,

we perhaps need not always blame our own stupidity but apparently rather our inertia on account of which the " momentum " fails to carry us on far enough.

One started by posing certain questions prompted by this book. There is another question one is prompted to ask about the book itself. Will it live as one of the great classics of science? Without any frivolous intentions, one might venture to forecast that it may live as *Robinson Crusoe*, but not as *Alice in Wonderland*. The latter lives in its author's own words, the former in countless re-tellings of the author's tale ; it is the *tale* which is immortal. And Eddington's tale has all the makings of immortality if only someone were found to re-tell it—in nursery language. W. H. McCrea.

The Royal Society. Newton Tercentenary Celebrations. Pp. xv, 92. 10s. 6d. 1947. (Cambridge University Press)

" The war prevented an International celebration, in 1942, of the 300th anniversary of the birth of Isaac Newton. The Royal Society of London took the earliest opportunity in July 1946 of inviting the national academies of science of the world to join with it in paying homage to his memory ". The present volume contains a record of the tercentenary celebrations, and prints in full the addresses given at the meetings. These are :

Address of welcome, by Sir Robert Robinson, P.R.S.

Newton, by Professor E. N. da C. Andrade, F.R.S.

Address of welcome, by the Master of Trinity, Dr. G. M. Trevelyan, O.M.

Newton, the Man, by the late Lord Keynes.

Newton and the Infinitesimal Calculus, by Professor J. Hadamard, For. Mem. R.S.

Newton and the Atomic Theory, by Academician S. I. Vavilov.

Newton's Principles and Modern Atomic Mechanics, by Professor N. Bohr, For. Mem. R.S.

Newton, the Algebraist and Geometer, by Professor H. W. Turnbull, F.R.S.

Newton's Contributions to Observational Astronomy, by Dr. W. Adams.

Newton and Fluid Mechanics, by Professor J. C. Hunsaker.

" Here's richness! " Here are the leaders of present-day scientific thought estimating for us the value of Newton's work both to his own age and to ours. Perhaps to-day a fairer estimate can be formed than has hitherto been possible, and yet the impression obtained from reading these papers is that passing years have not diminished the glory but increased it. Distance has made it easier for us to appreciate the colossal dimensions of Newton's genius, by enabling us to compare him with the other great thinkers of all time, and by allowing us to see how his thought is still potent after 250 years. Again and again, in these addresses from experts in widely differing fields, the same note occurs, the note of admiration at an unerring insight which at times appears to be more than human. Perhaps Lord Keynes' words (read to the delegates by Dr. Geoffrey Keynes) best sum up the general tone ; after referring to Newton's unequalled all-round technique, he goes on : " His *peculiar* gift was the power of holding continuously in his mind a purely mental problem until he had seen straight through it. I fancy his pre-eminence is due to his muscles of intuition being the strongest and most enduring with which a man has ever been gifted ".

The volume is beautifully produced, with three portraits of Newton, a picture of his rooms at Trinity, another of his birthplace, Woolsthorpe Manor, and a facsimile of a Newton letter to Halley. Moreover, though the papers are printed in the order in which they were given, this order is very convenient for the reader, beginning with Professor Andrade's masterly general sketch,

followed by the address by Lord Keynes referred to above, in which a brilliant study is made " of Newton *as he was himself* ", the best psychological study I have yet encountered, inspired by a sincere desire to see Newton as a man of the seventeenth century, not as the man the twentieth century may think he ought to have been. From there we can pass on to the more special studies, and it will be a lethargic reader indeed who finds it possible to put the book aside unfinished.

An inevitable choice both for the private collection and the school library, this volume ought at once to be added to the shelf which should already contain the Association's *Newton Memorial Volume*, edited by W. J. Greenstreet (1927).

T. A. A. B.

A Chapter in the Theory of Numbers. By L. J. MORDELL. Pp. 31. 1s. 6d. 1947. (Cambridge University Press)

This booklet represents Professor Mordell's Inaugural Lecture as Sadleirian Professor at Cambridge. He took as his subject the equation $y^2 = x^3 + k$, and its solubility in integers or in rational numbers. Professor Mordell justifies, both on historical and on personal grounds, his choice of a topic which might, at first sight, seem to be " insignificant or remote ". The history of the equation is certainly a curious one. New discoveries relating to it have been made at erratic intervals, from the time of Fermat onwards ; sometimes by means of a bright idea of a quite elementary character, sometimes by applications of general algebraic number-theory. These various advances are clearly outlined in the lecture, as far as that can be done without making too great demands on the reader's knowledge.

In the present century, work on this equation has provided a starting-point for new theories of great generality and importance. Our knowledge is still quite fragmentary, however, and much remains to be discovered. Professor Mordell has done a real service to the mathematical world by publishing this connected account of a topic on which he has been an acknowledged expert since the days of his Smith's Prize Essay.

H. D.

An Introduction to Analytical Geometry, Volume II. By A. ROBSON. Pp. 215. 10s. 6d. 1947. (Cambridge)

With his second volume Mr. Robson proceeds to those topics which, after Volume I, " seem to the author to be most worth a place in the early part of the geometry course ". And a comprehensive selection it is, ranging fully over the ground likely to be required for any reasonable Higher Certificate or Scholarship examination (up to 1949 at least!) and glancing at the vista beyond. A note of the chapter headings will give an idea of the contents : Homography ; Involution ; General Geometry ; Ranges on a Conic ; Systems of Conics ; Reciprocation ; General Cartesian Conic ; Foci and Confocals ; Normals and Evolutes ; Special Homogeneous Coordinates.

Mr. Robson uses, as always, whatever method comes most conveniently to hand, though one has the impression that " pure geometry " predominates. Cartesian and homogeneous coordinates are both in use, as well as polars, areals and trilinears. Many of the proofs, both of theorems and of examples, have that " slickness " which we expect of the author.

No two geometers seem to acquire quite identical outlooks [the outsider who regards mathematics as cut-and-dried would be astonished at the divergences] and I should not perhaps follow Mr. Robson at all points, but most of the differences would be on minor matters which need not concern us here. There is, however, one fundamental point which I raise with some diffidence in view of Mr. Robson's great experience in teaching at these levels. In Volume

I, the author sets out to distinguish very clearly between various geometries, named $G_1, G_2, \ldots, G_7$, ranging from Euclidean through Cartesian to the full complex projective geometry. In the present volume there are continued references to the different geometries, and I can imagine a schoolboy (and perhaps his teacher too) going delirious in the attempt (i) to remember which is which anyway; (ii) to spot which is being used at any particular moment. Thus, as far as I can see, the first two chapters of this volume (Chaps. 17, 18) do not define at any point precisely which space is being used; indeed, it would appear to be an oscillating space to cope with statements such as (p. 5) "in real geometry ...; and in complex geometry ...". Again, the first example on p. 13 is essentially metrical (with reference to "points at infinity") while the second would do equally well in complex projective space. It is not until the third chapter (Chap. 19) that we find the statement: "With the exception of some metrical interpretations the results of Chapters 17 and 18 can be made to apply to G_7 by small verbal alterations"—and I myself often make the smallness less than any ϵ! What I am afraid of is that the reader will become confused as he attempts to move from geometry to geometry. It is possible that the attempt to produce precision has involved the introduction of more variety than is appropriate at this level.

But let us honour Mr. Robson for tackling boldly what many writers have preferred to shelve. The book as a whole must give a stimulus to geometrical teaching at many points.

E. A. Maxwell.

Foundations of Algebraic Geometry. By André Weil. Pp. xix, 288. \$5.50. 1946. (American Mathematical Society Colloquium Publications, Vol. 29)

Algebraic geometry is that branch of mathematics which deals with the geometrical interpretation of algebraic equations. It has been universally recognised as an important and attractive branch of mathematics, but many mathematicians have been prevented from cultivating it by the feeling that its principles and methods are only fully understood by a small number of people, and that the novice wishing for initiation must undergo a long training in the avoidance of clear-cut (if laborious) algebraic methods, and in the development of his geometrical intuition.

Critics of the methods used for demonstrating theorems in algebraic geometry have been quick to note that although geometrical intuition is a valuable tool, those possessed of it have sometimes disagreed with each other about the validity of certain results obtained with the help of this tool. Impartial outsiders have sometimes resolved these discords, but only by constructing an analytical proof which is verifiable by anyone with a normal mathematical equipment.

Algebraic geometry has always had foundations, algebraic ones in fact, but it is only recently that anyone has endeavoured to develop the subject from its fundamentals, in a way which can be understood by all competent mathematicians. A pioneer in this development is B. L. van der Waerden. His book, *Einfuehrung in die Algebraische Geometrie*, is still obtainable in a photographic reprint. It may be compared with some of the standard Italian works, such as Enriques-Chisini's *Teoria geometrica delle equazioni*, Bertini's *Geometria proiettiva degli iperspazi*, and Severi's *Trattato di geometria algebrica*. It will be seen that van der Waerden, in crystallising some of the concepts latent in Italian geometry, has produced new and important ideas of his own. It is only necessary to mention (1) proper specialisation, (2) the generic point of an algebraic variety, (3) the Cayley (zugeordnete) form of an algebraic variety.

But there is still a good deal to be done before it can be said that every result described in one of the three Italian books mentioned can be proved rigorously. On the other hand, new methods breed new theorems. Zariski, using the methods of ideal-theory introduced, but later abandoned, by van der Waerden, has developed the theory of birational transformations to a considerable extent, deepening and proving known theorems and discovering many new ones. It can now be said, for instance, that any irreducible algebraic surface can be birationally transformed into one without singularities. This theorem has been demonstrated many times. But so far only Zariski's proof has won universal approbation. The ultimate aim of workers on the foundations of algebraic geometry is to erect an aesthetically pleasing structure, free from logical faults, on which the many ornaments of Italian geometry can be tastefully displayed. Such a structure will carry its own ornament, and in the course of time the Italian decorations may well become only a secondary feature of the building.

So far we have been discussing classical geometry, which is over the field of complex numbers. We may expect the theorems proved in this geometry to be true over any field of characteristic zero. If one wishes to investigate algebraic geometry over fields of characteristic p, a completely new research has to be undertaken. André Weil explains in his preface that his book arose " from the necessity of giving a firm basis to Severi's theory of correspondences on algebraic curves, especially in the case of characteristic $p \neq 0$ (in which there is no transcendental method to guarantee the correctness of the results obtained by algebraic means), this being required for the solution of a long outstanding problem, the proof of the Riemann hypothesis in function-fields ". It is emphasised that " the main purpose of the book is to present a detailed and connected treatment of the properties of intersection-multiplicities, which is to include all that is necessary and sufficient to legitimise the use made of these multiplicities in classical algebraic geometry, especially of the Italian school ".

The chapter headings are : I. Algebraic preliminaries. II. Algebraic theory of specialisations. III. Analytic theory of specialisations. IV. The geometric language. V. Intersection-multiplicities (special case). VI. General Intersection-theory. VII. The geometry on abstract varieties. VIII. Functions and divisors. IX. Comments and discussions. Appendix I. Projective spaces. Appendix II. Normalisation of varieties. Appendix III. Characterisation of the i-symbol by its properties.

We cannot tell how this book will strike the non-specialist reader. In appearance it is rather formidable, its 290-odd pages being very closely packed. The author naturally makes use of some of the fundamental concepts introduced by van der Waerden, but generalises them so extensively that we wonder whether anyone unfamiliar with van der Waerden's work will see the point of the definitions given in Chapter IV. Again, it is always rather dangerous, without good reason, to break away completely from a normal historical development, and when we read in a footnote " the device which follows, it may be hoped, finally eliminates from algebraic geometry the last traces of elimination-theory . . . ", we feel that things have been made very hard indeed for anyone wishing to understand the subject.

But after these criticisms it must be said that the book is beautifully written, and a remarkable piece of mathematics. The author has gone to great trouble to guide the reader, to explain what he is doing at every stage, and finally brings him, slightly dazed perhaps, to the frontier of present-day knowledge in certain branches of algebraic geometry. This is obviously a book which deserves and will repay careful study. D. PEDOE.

A Brief Course in Analytic Geometry. By P. P. Boyd and H. H. Downing. Pp. 180. 14s. 1947. (D. Van Nostrand and Macmillan)

This short course by two professors in the University of Kentucky is an elementary account of analytical metrical geometry of two and three dimensions. It deals with the point, line, circle, conics referred to their principal axes, and a few transcendental curves, and, in the last third of the book, with the plane and the quadric surfaces referred to their principal axes.

The coordinates used are naturally enough for the most part rectangular cartesian, but polar coordinates are also explained and well illustrated. There is a useful emphasis on the equation of a locus, and also on the complementary idea which the authors call the "locus of an equation". Chapter IX on parametric equations comes at the end of the two-dimensional section, and so parameters are not used as much as is now thought desirable in dealing with the conics. These curves are treated mostly from the point of view of the standard equations

$$y^2=4ax,\ x^2/a^2\pm y^2/b^2=1,\ (y-k)^2=4a(x-h),\ (x-h)^2/a^2\pm(y-k)^2/b^2=1,$$

and $r=ek/(1-e\cos\theta)$. It is good to find mention of such curves as the trigonometrical graphs, $y=\log x$, $y=\exp x$, and the lemniscate, limaçon, spirals; also in chapter IX the hypocycloid and epicycloid.

Solid geometry is started in the usual way. Spherical polar and cylindrical coordinates are included, and direction cosines are used. The line in three dimensions is represented by two linear equations, the equations of planes through it. The idea of duality is not used, and though this may be inevitable in such an elementary sketch of two-dimensional work, it would seem strange in England that the idea should not be in evidence by the time that solid geometry was being studied. For example, the dual idea of a line regarded as determined by two of its points instead of by two of its planes is important. Quadric surfaces are introduced in the order cylinder, sphere, cone, and then

$$x^2/a^2\pm y^2/b^2\pm z^2/c^2=1 \text{ and } x^2/a^2\pm y^2/b^2=z.$$

The lines on the ruled quadrics are obtained, and the cylindroid occurs as an example. The use of parameters for twisted curves is illustrated by the circular helix.

The problems are arranged in two general groups: those that the authors call the moderately difficult ones are grouped in threes, and the generally more difficult ones are placed in miscellaneous lists at the ends of the chapters. Very few of these deserve the name of "problem" or the epithet "difficult". No doubt the course is intended to be thoroughly easy and elementary. In England these subjects are probably taken only by small numbers of pupils (and by those at schools, not universities), but so far as the solid geometry is concerned only by pupils with special mathematical ability; and for them this introduction to solid geometry would not be satisfactory, because they need to learn particularly about those matters in which three-dimensional geometry differs from two-dimensional, and they are usually more sophisticated about parameters, duality, and the projective standpoint than prospective readers of this short course in an American university. For the purpose for which it is intended, the book will probably prove very suitable. It is very well produced, and contains answers and a good index. A. R.

Functions of a Complex Variable. By T. M. MacRobert. Third edition. Pp. xv, 390. 18s. 1947. (Macmillan)

The appearance of a third edition of this well-established book on functions of a complex variable will be welcomed. In the second edition, four appendices

and a second set of miscellaneous examples were added. In the present edition, the work is further enlarged by the addition of a fifth appendix and a third set of miscellaneous examples. The new appendix is in effect a chapter of twenty pages on generalised hypergeometric functions, upon which many papers have been written during the past thirty years. The author is to be congratulated on finding space for a substantial introduction to this comparatively recent off-shoot of mathematics in a standard book, where it will no doubt awaken the interest of a wider circle of readers. They will find a bibliography up to 1935 in the Cambridge tract on generalised hypergeometric series by W. N. Bailey. The new set of miscellaneous examples, with many hints to meet difficulties, occupies twenty-two pages more.

These considerable additions have naturally called for an increase in price, but the increase is a small one. Paper and print appear to be as good as ever.

Where good value is offered at small cost, it would be unreasonable to ask for more, but it may not be out of place here to draw attention to the absence of an introductory book in English on algebraic functions of a complex variable, less formidable than Baker's large volume, of the same scope as Appell and Goursat's *Fonctions Algébriques* or Landfriedt's *Algebraische Funktionen*. F. B.

Differential and Integral Calculus. By F. D. Murnaghan. Pp. x, 502. 1947. (Remsen Press, New York)

There is no clear-cut distinction between the *Calculus* and the *Theory of Functions* but these titles serve roughly to distinguish the mere technique of differentiating and integrating from the underlying foundation theorems. In this sense it is the Theory of Functions rather than the Calculus which forms the content of the greater part of Murnaghan's text, but I think the title " Calculus " was deliberately, and rightly, chosen to indicate, not the content but the character of the book and the class of students for which it was written. It is in the training of the applied mathematician, the physicist and the engineer, not the mathematical specialist, that Murnaghan is interested, and the importance of the book lies in the rigour and severity of an account designed for this purpose.

Recent years have brought a growing realisation that an ability to differentiate and integrate without an understanding of the inner significance of these operations leaves a scientist the servant and never the master of the " language of nature ". Even those who cannot accept the extreme position taken in Jeffreys' *Methods of Mathematical Physics*, that mathematical rigour is more important to the physicist than to the mathematician himself, will concede that the depth and breadth of the present-day applications of mathematics impose almost as severe a mental discipline on the student of physics as a study of the elements of analysis.

In the *Differential and Integral Calculus* the author has done everything that humanly could be done to make the classical theory of functions of a single variable comprehensible to a beginner. Difficult arguments are broken up into several stages, often separated by carefully designed illustrative exercises. The treatment throughout is extremely thorough, and no detail is overlooked. The development is rigorous and exact, and although the free play of intuition is encouraged by numerous diagrams, it is never part of the argument that " it is evident from the figure ".

In most respects the book proceeds along traditional lines. A real number is defined by a nest of rational intervals and an equivalent of Dedekind's theorem is proved in the form : every nest of intervals (rational or not) covers a unique real number. The arithmetic of real numbers is studied in some

detail ; upper and lower bounds of a variable are defined, and the limit of a function at a point (though upper and lower limits are not introduced). The differential calculus, with the emphasis on *differentiability* rather than on the *derivative* itself, is developed up to the Extended Mean Value Theorem (with Lagrange's remainder). There is a very thoroughgoing and detailed account of the Riemann integral, followed by a quite adequate summary of the integration processes for combinations of elementary functions.

An explicit definition of *angle* is given in terms of the theory of rectifiable curves and the circular functions are derived by inverting the relation between the length of an arc of a unit circle and the x-coordinate of the variable end-point of the arc. Attractive though this method is, its drawback is that the periodicity of the circular functions must be postulated, just as in the analogous method of defining arc cos x by the integral of $1/(1-x^2)^{\frac{1}{2}}$.

In a book of 500 pages the omission of Cauchy's Mean Value Theorem for a pair of functions is rather striking ; as a consequence of this omission the proof of the Extended Theorem of the Mean is unnecessarily complicated, and of course we lose L'Hospital's theorem on the limit of a quotient. In view of the importance the author attaches to computation, another surprising omission is Stirling's formula.

In a few minor points only does the treatment fail to maintain its very high standard of clarity, thoroughness and accuracy.

(i) On page 26, proof is lacking that the same relation holds between two rational real numbers which holds between the rationals to which they correspond.

(ii) The account of Mathematical Induction in the Review Exercises on page 47, at the end of Chapter I, is misleading.

(iii) An unusually good description of differentials is partly vitiated by the needless complexity of the opening paragraphs.

(iv) On page 90 it is wrongly stated that the theory necessary to prove the decisive theorem on maximum and minimum values depends upon the Mean Value Theorem. The necessity for, and sufficiency of, the familiar conditions upon $f^n(a)$ for a maximum or minimum value of a repeatedly differentiable function $f(x)$ at $x=a$, are readily proved without appeal to any existence theorem.

(v) The proof of the continuity of the inverse of a monotonic continuous function, on page 137, appears to be inadequate. Moreover, it is not true (page 138) that the relationship between a function and its inverse is a *partnership*, unless the functions are monotonic.

(vi) There is an excellent " natural " proof of the formula for substitution in the definite integral (for the case of a monotonic substitution function), which does not assume the continuity of the integrands, but it is wrongly suggested, on page 315, that the substitution formula is valid only when the substitution function is monotonic.

(vii) The section on Partial Fractions is unworthy of the book ; the treatment is archaic and even Hermite's rule is not mentioned.

Throughout the book the term *function* is confined to single-valued functions. Consequently a circle is not the graph of a function ; instead we have upper and lower semicircles defined by $y=+\sqrt{(a^2-x^2)}$, $y=-\sqrt{(a^2-x^2)}$ respectively, and the reader is expressly warned against the statement (a warning which I gladly endorse) : the circle (with equation $x^2+y^2=a^2$) is the graph of the two-valued function $y=\pm(a^2-x^2)^{\frac{1}{2}}$. Murnaghan accepts the current definition of function in terms of dependent and independent variables, the term variable connoting a class of numbers. One variable is said to be a function of another if with each and every member of the second variable a unique member of the first is associated in an unambiguous manner. The

weakness of this definition lies, of course, in the vagueness of the condition "associated in an unambiguous manner", since *associated* is undefined; and it would certainly seem preferable to define *association* in terms of *function*, rather than conversely, accepting, at any rate in an elementary text, an "evolving" definition of function (for instance, xy and $x+y$ are functions, and if $f(x)$, $g(x)$ are functions so are $f(x)g(x)$, $f(x)+g(x)$ functions, and so on). It is only in the case of inverse functions that we are presented with an actual correlation of variables, and even in this case the correlation is established by a previously defined function. If we followed this procedure we could, with advantage, dispense with the dual terminology of *function* and *dependent variable*, and so confine the term variable to the independent variable, and render unnecessary Murnaghan's well-pointed warning that a variable may be a constant.

The book is extremely well printed in a clear bold fount, with well-spaced pages and ample variations in type. Both the author and the printer are to be congratulated on a book which merits the attention of all University teachers of elementary analysis. A companion volume on functions of several variables is promised shortly.

Misprints: p. 51, l. 12, for "y'" read "y,".
p. 67, Exercise 2, for "$y(x) = -1$ if $x=0$" read "$y(x) = -1$ if $x<0$".
p. 449, 3 lines from bottom of page,
for "$M_\kappa' > M_\kappa' + \epsilon$" read "$M_\kappa > M_\kappa' + \epsilon$".
p. 450, l. 2, for "M" read "M_κ'". R. L. GOODSTEIN.

Advanced Calculus. By D. V. WIDDER. Pp. xvi, 432. 36s. 1947. (Prentice Hall, New York; Constable, London)

The student of applied mathematics who learns his Advanced Calculus from this book will not only find in Analysis a superlative instrument, but will gain a very deep appreciation of the beauty in a structure of pure reason. A graceful and smooth flowing style, a mastery of clear exposition, generous and expansive lay-out and printing all combine to make the reader's lot a happy one.

Advanced Calculus is not intended for the mathematical specialist in training, for there is no emphasis on generality. Each theorem is established under the simplest conditions adequate for application. Hypotheses and conclusions are numbered and listed, and what is proved, and what assumed are apparent at a glance. The treatment and development follow orthodox lines, but the emphasis on a continuous derivative effects considerable simplification.

Simplification is not, of course, without its companion dangers. The mean-value theorem is stated in the text for a function differentiable in a closed interval (a, b), but the application made on p. 219 to prove

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

requires a mean-value theorem established under the traditional conditions of continuity in the closed interval (a, b) and differentiability in the open interval (a, b), since $f(1/x)$ is not differentiable at the origin. The same objection applies to the proof of theorem 4 on p. 221.

The mean-value theorems are carefully stated to bring out their character as existence theorems, with the exception of the integral mean-value theorem on p. 34, where the equation

$$\int_a^b f(x)\,g(x)\,dx = f(X)\int_a^b g(x)\,dx$$

is given the appearance of holding for all X in (a, b).

The corresponding formula for a Stieltjes integral, on p. 137, is equally misleading.

One of the features of the book is the derivation of familiar formulae of applied mathematics from axiomatic definitions of physical concepts; there are no "little boxes" or line segments behaving like points, as in the old tradition of potential theory. Integral formulae are obtained from inequalities expressing the defining characteristics of the concept, and the argument is no harder to follow than the mischievous swindle that is so often thought good enough for applied mathematics.

In a book which commences, so to speak, half-way through the Calculus, with partial differentiation, it is sometimes difficult to know what the author assumes as known. We are not told, for instance, the number system on which the whole work is based, and though most of the fundamental theorems, like the Heine-Borel in one dimension, are proved in the course of the book, we are sometimes kept waiting for a warning that a certain argument needs subsequent implementation (*e.g.* on pp. 146-147). It is assumed without proof on p. 16 that $f'(x)=0$ throughout (a, b) implies $f(x)$ is constant in (a, b); since the mean-value theorem on which this result depends (in the traditional development) is proved in the text, it is hard to accept this result as something supposed already known.

In the first chapter two points which deserve further consideration are the definition of $\lim\limits_{x\to\infty} f(x)=A$ (which is left as an exercise to the reader, following the definition of $\lim\limits_{x\to a\pm} f(x)$) and the definition of a limit point.

The definition given (p. 10) requires that every neighbourhood of a limit point l of a set S *contains points of* S. This geometrically-worded definition is open to serious objection. In the first place any neighbourhood of an *isolated* point l of a set S contains points of S (namely l itself). If we attempt to meet this difficulty by adding the qualification that every neighbourhood of a limit point contains points of S other than l, then we exclude the important case "for all n, $s_n=l$, implies $\lim s_n=l$". On the other hand if we require that every neighbourhood of a limit point should contain at least two elements of the set, then we may be unable to proceed without appeal to the Multiplicative Axiom. Simplest perhaps is the explicit disjunction that l is a limit point of a set S, if $x=l$ for infinitely many elements x of S or corresponding to any assigned neighbourhood of l there is an element x of S contained in the neighbourhood, $x\neq l$.

The book could with advantage dispense with the account of upper and lower limits, since only the most trivial applications are made of them.

The chapter on differential geometry deals adequately with the elements of curve and surface theory, but the notation employed for vector products is a very unhappy one, and particularly disappointing after the excellent notational innovations in the first chapter. One other point of notation that seems somewhat unsatisfactory is the expression of the proposition "$f(x)$ is continuous at $x=a$" by "$f(x)\in C$, $x=a$", where C denotes the class of continuous functions. It is the class of functions continuous at $x=a$ to which $f(x)$ belongs; a function cannot be said to belong to a certain class of functions, at some one point.

In the account of saddle points, the statement (p. 107) that it is not sufficient for the existence of a minimum that the homogeneous polynomial of lowest degree (>1) in the Taylor expansion should be always positive, seems to be mistaken. The example under consideration is

$$f(x, y)=y^2-x^3,$$

and since the term of second degree may vanish without x vanishing, there is no minimum at the origin by the ordinary criterion.

A pleasing chapter on the theory of envelopes successfully avoids the fallacies so common in elementary accounts by means of an explicit definition of the envelope. A family of curves $f(x, y, \alpha) = 0$ is said to have an envelope

$$x = g(\alpha), \quad y = h(\alpha) \qquad \text{(E)}$$

if, and only if, for each $\alpha = \alpha_0$ the point $(g(\alpha_0), h(\alpha_0))$ of the curve (E) lies on the curve $f(x, y, \alpha_0) = 0$, and both curves have the same tangent there.

Chapter IX on infinite series contains no reference to the *interval of convergence*, but a knowledge of this interval is assumed later on p. 366, and its existence is proved on p. 369. Amongst the convergence tests described,

$$\lim_{k \to \infty} k^p u_k = A, \quad p > 1,$$

is mentioned as a somewhat neglected test, but of course it is only a special case of the widely-used comparison test $u_k/v_k \to A$. A common fallacy in the integration of series makes its appearance on p. 260, where it is wrongly maintained that the convergence of $\Sigma u_k(x)$ for $a \leqslant x \leqslant b$ (as well as the uniform convergence of $\Sigma u'_k(x)$) is *necessary* for term-by-term differentiation. In fact we require to know only the convergence of $\Sigma u_k(x)$ at a single point of the interval (a, b), for if $\Sigma u_k(x)$ converges at $x = a$, and if $\phi(x) = \Sigma u'_k(x)$, then $\int_a^x \phi(t)\, dt$ exists and equals $\sum_0^\infty \{u_k(x) - u_k(a)\} = \lim \{s_k(x) - s_k(a)\}$, whence, since $\lim s_k(a)$ exists, $\lim s_k(x)$ exists for any x in (a, b).

Two unfortunate omissions from the valuable chapter on the convergence of improper integrals are consideration of *substitution* in, and *differentiation* of, improper integrals. The chapters on the Gamma function and the Laplace transform introduce both these operations without comment.

The concluding chapters on Fourier series and the Laplace transform are particularly attractive. Of many elegant details one I cannot forbear to mention is the derivation of Weierstrass' theorem on polynomial approximation from Fejér's proof of the Cesàro summability of a Fourier series.

R. L. GOODSTEIN.

Theory and Application of Mathieu Functions. By N. W. McLACHLAN. Pp. xii, 401. 42s. 1947. (Oxford University Press)

The need for a comprehensive book on Mathieu Functions has for some time been felt by mathematicians and technologists, and Dr. McLachlan's book, though addressed to the technologist only, will be welcomed by many outside this group.

In studying the problem of the vibrations of an elliptical membrane Emile Mathieu in 1868 was led to the investigation of the equation

$$\frac{d^2y}{dz^2} + (a - 2q \cos 2z)y = 0, \qquad (1)$$

and this original form of the differential equation is accepted by the author as the canonical form of the Mathieu Equation. Mathieu himself developed the periodic integral order solutions of (1) in the form of infinite series of trigonometric functions and gave explicit expressions for the characteristic numbers in terms of q, valid for small values of this parameter. Corresponding to any point in the (a, q) plane there is always a solution of (1), but the periodic integral order solutions are particularly related to the special sets of points

which lie on characteristic curves. The regions in the (a, q) plane lying between the characteristic curves are called stable or unstable according as the corresponding solution of (1) is of the form $\exp(i\beta z) \times$ Periodic Function or $\exp(\mu z) \times$ Periodic Function. The (a, q) plane so patterned is called the Stability Chart. These aspects of Mathieu Functions are included in Chapters II–IV, while Chapter V together with sections of earlier chapters show, with carefully worked examples, how the characteristic numbers, the coefficients of the Mathieu Functions and the important Floquet numbers β and μ may be computed for any range of the parameter q. When the first solution of equation (1) is chosen to be the periodic integral order solution the second linearly independent solution must be non-periodic; this demonstration, and a complete discussion of the method of computation of the second solution, occupies Chapter VII. The alternative form of solution of (1) in an infinite series of Bessel Functions, and of products of Bessel Functions in Chapters VIII and XIII, the asymptotic expansion of the solution of (1) for large z and for large q in Chapter XI, and the distribution of zeros in Chapter XII form a well-coordinated group in the theory. The reader, however, who expects the miscellaneous integrals to be of the comprehensive and complex variety found in Watson's *Bessel Functions* will be disappointed with Chapter XIV. No occasion has arisen in the book which necessitates introducing the recurrence formulae for Mathieu Functions, but, in view of their latent possibilities, they might profitably have been included.

Chapters XV–XIX are concerned with the applications of Mathieu Functions. In physical problems Mathieu's Equation enters naturally in two ways:

In the first place the equation arises in the discussion of the oscillations of a system subject to forces which have a time periodic or space periodic variation. This is the case in most of the applications in Chapter XV. In the displacement problem of the moving coil loud speaker a demonstration is given of the application of the stable solution. In the application to the frequency modulation problem, the capacitance in a circuit provides the periodic variable, and it is shown that the time variation of the quantity of electricity in the capacitance is governed by an equation approximately of the Mathieu type; here also an interesting application is made of the Stability Chart in discussing the possibility of instability in a circuit.

In the second place Mathieu and modified Mathieu Functions arise in the two-dimensional problems involving the field equation $(\nabla^2 + k^2)\phi = 0$, the field boundaries being elliptic in form. Chapters XVI–XVIII contain numerous examples of this type of problem, the boundary conditions varying slightly in the various applications. Mathieu's elliptical membrane problem and the oscillations of water in an elliptic lake are discussed in some detail, since numerical results are available for these. General solutions are given for the flow past an elliptic cylinder of viscous fluid (although it is not stated that Mathieu Functions may also be conveniently applied to the flow past a long cylinder of arbitrary cross-section), for electrical and thermal diffusion from elliptic cores, elliptical wave-guides and diffraction around an elliptic cylinder.

The author justly stresses the need for the computation of the Floquet numbers and the modified functions, and to extend the Tables already prepared by E. L. Ince for the characteristic numbers and Mathieu Functions themselves.

The book is clearly written, contains a comprehensive list of references and will certainly meet the requirements of those to whom it is addressed. It is hardly necessary to comment on the high quality of the printing as the book is produced by the Clarendon Press. T. V. D.

Statistics. By L. H. C. Tippett. Pp. 184. 3s. 6d. net. 1945. No. 156 of The Home University Library of Modern Knowledge. (Geoffrey Cumberlege, Oxford University Press)

This little book is probably well known to many teachers. It does not appear to have been reviewed earlier in the *Math. Gazette*, and the publishers have now sent us a copy of the 1945 reprint of it: it was first published in 1943, and already previously reprinted in 1944. It is thus clearly popular: and deservedly so. Mr. Tippett, Statistician to the British Cotton Industry Research Association, is well known as the author of *The Methods of Statistics* (Williams and Norgate, First Edition, 1931: revised, 1937: reviewed in the *Math. Gazette*, XVI, 157). The Home University booklet is non-mathematical and almost non-arithmetical, but gives a clear idea of some of the fundamental notions of statistics, its procedures and its applications. Thus there are explained the ideas of mean, variation and skewness in connection with frequency distributions, contingency tables, correlation diagrams, random and biassed sampling and standard error, and references are made to analysis of variance, though not by name (p. 75), design of experiments (p. 103), and quality control, (again not by name: p. 150), and to various applications such as analysis of accident proneness.

The book has been found useful in, we believe, more than one sixth form for a simple outline of an elementary course for non-mathematicians, and it can be cordially recommended to teachers of such work as an easy reading book for these purposes. Frank Sandon.

Proceedings of the First Canadian Mathematical Congress, Montreal, 1945. Pp. xliv, 367. $3.25. 1946. (University of Toronto Press)

These proceedings give the impression that the first Canadian Mathematical Congress was a great success, and that it will be followed by many further meetings, by means of which the development of mathematics in the Dominion will be considerably stimulated.

The working programme of the Conference can be divided into three parts: lectures, short research notes, discussions. The lectures form an attractive section of the volume, but it is impossible to deal fully with them, and we must be content to suggest their range and value by a casual selection of names and topics: Brauer (algebra); Coxeter (regular solids); Mordell (geometry of numbers); Tucker (topology); Birkhoff (algebra); Hartree (the differential analyser); Chevalley (Lie groups).

The four discussions should interest every reader of the *Gazette*, since they deal with problems of teaching and organisation of universal importance: Secondary School Mathematics; Statistics; Engineering Mathematics; Research and Graduate work. Naturally the background is in each case Canadian, but much of what is said is of general application. In the discussion on secondary school mathematics one paragraph of Professor Norman Miller's contribution may give cause for thought in our own Association. He is discussing Isolationism in Canadian education and urges the formation of regional groups of teachers. One purpose of these groups, he properly suggests, would be to subscribe collectively to such journals as the *Mathematics Teacher*, the *American Mathematical Monthly*, and to such reports as the *Yearbooks* of the U. S. A. National Council of Teachers of Mathematics. He further advocates affiliation with the American societies. The success of our Association's Branches in Australia and New Zealand, and the high standard of our Teaching Committee's *Reports*, are evidently little known in Canada. Is that our fault? If so, it should be promptly remedied. T. A. A. B.

The Theory of Potential and Spherical Harmonics. By W. J. STERNBERG and T. L. SMITH. Second edition. Pp. xii, 312. 20s. 1946. Mathematical Expositions, 3. (University of Toronto Press; London, Geoffrey Cumberlege)

This new edition is effectively a reprint, and we may refer readers to the comprehensive notice of the first edition by Professor Copson (*Math. Gazette*, XXIX, pp. 34–36). The series of which it is a member is now well recognised as a valuable addition to textbook literature at the university level. Readers should note that the British agent is Geoffrey Cumberlege, Oxford University Press.

T. A. A. B.

The Foundations of Geometry. By G. de B. Robinson. Second edition. Pp. xi, 168. 14s. 1946. Mathematical Expositions, 1. (Toronto University Press; London, Geoffrey Cumberlege)

The first edition of this book was reviewed in the *Gazette* (XXV, p. 186), and the appearance of the second edition, so soon after, is a testimony to the author's skill and one of the few signs of a return to civilisation. The topics treated centre round the work, now classical, of Hilbert and his contemporaries, at the turn of the century, and the reviewer would select for special attention the introduction of coordinates.

It must always be a matter of astonishment at first sight that, for example, *if*, in a plane, we assume only the basic projective axioms that two points fix their unique joining lines and two lines their unique point of intersection, together with Pappus' theorem, *then, without any use of the notion of distance*, we can assign to each point two coordinates, which are elements of a field (that is, a collection of elements satisfying the ordinary formal laws of addition and multiplication), and that, with these coordinates, a line has a linear equation.

If instead of Pappus' theorem we assume Desargues' theorem on perspective triangles, our coordinates still satisfy the formal laws of algebra except the commutative law of multiplication. Thus of the two chief actors, Desargues is less powerful than Pappus.

Much the same is true for projective space of three and more dimensions, except that now Desargues' theorem (but not Pappus') follows from the basic projective axioms, and need not be adjoined to them. Finally, in the plane, Desargues' theorem follows from Pappus', but not *vice versa*. The relation between these geometric facts and the corresponding algebra, and the effects of assuming continuity or the axiom of Archimedes, are absorbing topics to be found in this book.

But we can go further. In a descriptive geometry, two coplanar lines may or may not meet. If the geometry is three-dimensional and satisfies the ordinary order axioms, it can be "made projective", and this gives us a still more surprising instance of the introduction of coordinates.

There have been developments in recent years. Ruth Moufang in some unnecessarily complicated papers has considered that case of Desargues' theorem which is equivalent to the assumption of the uniqueness of the fourth harmonic point (given three collinear points), and has shown that this gives all the formal laws of algebra for the coordinates, except, of course, the commutative law of multiplication, and except that the associative law of multiplication can only be shown when two of the three factors are equal. We have the "alternative" algebra, now so well known to algebraists. Further analysis was suggested by Blaschke's *Textile Geometry*, and this was carried out to some extent by Reidemeister.

All these are refinements of the earlier results of which this book gives such an excellent account.

H. G. F.

Mathematics as a Culture Clue. By C. J. KEYSER. Collected Works, Vol. I. Pp. 277. $3.75. 1947. (*Scripta Mathematica*, Yeshiva University, New York)

Here is collected a series of articles, many dealing with logical questions of the *Principia* era. It is difficult to determine the original dates of publication, but in two articles which appear to be comparatively recent, I was surprised to find a statement that " P implies T " asserts that " T is logically deducible from P ".

The best thing in the book is a notice of C. S. Peirce, with copious quotations from his writings. I give a quotation for the reader to consider. " It is a matter of fact that there are four eyes in the room. But to say that *if* there are two persons and each person has two eyes there *will be* four eyes is not a statement of fact, but a statement about the system of numbers which is our own creation ".

Other accounts deal with the mathematical economist Pareto and with W. B. Smith, the mathematician who supported the mythic theory of the origin of Christianity and wrote *Der Vorchristliche Jesus* (1906) and *Ecce Deus* (1911). H. G. F.

The Magic of Numbers. By E. T. BELL. Pp. viii, 418. 17s. 6d. 1946. (McGraw-Hill)

When a mathematician has found a proof for some theorem which may have appeared new and unexpected, it will often seem to him as if he was only bringing to light something which had been in existence all along, which would have been true even without his own proof. On another occasion he will be conscious of an intricate constructive effort while piecing together varied bits of previously acquired knowledge in order to build a proof. On both occasions the intensity and immediacy of his experience may tempt him to an obvious general creed about the nature of mathematical knowledge, its *discovery or invention.* The scientist who thinks out a mathematical theory to comprehend his experiments and observations finds that mathematical conclusions drawn from this theory will accurately predict the outcome of some subsequent observations. He too may be tempted to build a philosophy from this experience and to try and find Reality on either side—or at least to *get clarity on the meaning* of mathematical ideas, *of Numbers* in particular, and on the *scope and reliability of deductive reasoning.* Bell gives an account of the history of this eternal dilemma from Pythagoras and Plato to Hardy and Eddington. It reads like a racy historical novel, full of dramatic life-story detail, full of witty, sarcastic remarks and innuendos; it often reads merely like a (somewhat uncharitable) History of Human Errors—errors due to the timeless desire not only to know but also to understand. Historians are often admired for their ability to create the contemporary atmosphere in their description of a certain period of the past. Bell's technique is exactly opposite, he brings the atmosphere of his own environment into the past and surely makes some of his readers feel at home with, say, the *rascal Thales who cornered the market in oil-presses* and Plato who *peddled oil in Egypt.* More seriously, this attitude dissolves the theory of Proportions into empty Fractions, for the anachronistic benefit of the *normal child of twelve* of our time, to give but one example. History is, of course, always seen backwards, and easily wise after the event. Nowhere is this so dangerous as for the historian of Science : because Science *does* progress, whatever the history of wars, politics, economics, of philosophy even, may have to record. Present knowledge of science fits all the facts, including " old " facts, and so the modern historian is easily given to praise, condemnation, mockery from his Western point of view. One gladly partakes in the author's lively personal interest when he finds things *amusing, depressing, enervating,* or *trivial to us*—again, the reader must be accustomed to use the

term Western Civilisation in the sense of Material and Technical Civilisation; he is not expected to be familiar with either the Gospels or Hamlet; he ought not to want to read Plato in order to find an alleged quotation; he must on occasion be content to have some historical claim substantiated by simple repetition; he should remember his Undergraduate Society debating the eternal "Hen-or-Egg" Priority problem; but before all else he must be mature enough not to be muddled, as so many of us are, by irony and sarcasm —and he must certainly not be pedantic about detail. He will like and enjoy the book greatly for its stimulating, provoking, exasperating style—he will be thankful to be urged to think for himself again, rather desperately at times, to keep, or to find, his own bearings in this timeless conflict of evidence and reasoning, intellect and insight, knowledge and vision, experiment and pure thought; laying aside, if possible, his own not quite so far Western idiosyncrasies. And if he happens to be a schoolmaster, he will gladly accept for his own use the line, "What in the name of Zeus is a hypothenuse?"

A. P.

Guide to the Literature of Mathematics and Physics. By N. G. Parke. Pp. xv, 205. 25s. 1947. (McGraw-Hill)

Discriminating critics, with friendly candour, have noted as defects of English mathematics its amateurism, its insularity, its lack of systematic thoroughness. English readers of Mr. Parke's book should bear this criticism in mind, for otherwise they may feel undue impatience over his occasional concern with niggling detail. The author is trying to help the many who need to know "where to find and how to assimilate information"; and he has set about his task in a sensible and praiseworthy fashion.

The book is in two parts. The first part, "General Considerations", has four chapters: Principles of Reading and Study, Self-directed Education, Literature Search, Periodicals. The author is well enough aware that his suggestions are by themselves no royal road, but they may help the novice to find some kind of path through the enormous and rapidly increasing mass of treatises and periodicals dealing with mathematics and mathematical physics. Naturally the guidance is concerned chiefly with U.S. literature, and library classifications are related to the Library of Congress arrangements. I am not familiar with the "Union List of Serials", giving the periodical holdings of the main U.S. and Canadian libraries, but the author speaks of its value, and those who have had the tantalising experience of being held up in obtaining a promising article by inability to find a library containing the journal required, will know how valuable such a list can be. Not the least of Greenstreet's services to mathematics was his *Catalogue of Current Mathematical Journals with the names of Libraries in which they may be found* (1913): and it is no credit to this country that so useful a list has not been brought up to date and re-issued.

The second half of the book lists over 2000 treatises on mathematics and physics, under about 150 alphabetically arranged subject headings. The list is not systematically annotated, but a certain amount of comment on subjects or specially valuable items is supplied. "Chance and the author's personal bias play a strong role" but there is no evidence of unfairness or prejudice.,

T. A. A. B.

Introduction mathématique aux théories physiques modernes. I. Nombres complexes, nombres hypercomplexes, matrices, opérateurs, applications élémentaires. Par M. Morland. Pp. 139. 350 fr. 1947. (Vuibert, Paris)

The underlying idea of this book would seem to be the introduction of the various topics mentioned in the title in relation to groups of geometrical transformations. Thus, negative numbers are derived from positive ones by

means of the group of translations along a line. The group of translations in a plane introduces column vectors :

$$\left\| \begin{matrix} a \\ b \end{matrix} \right\|$$

and the group of rotations about a fixed point leads naturally to matrices of the type :

$$R(\theta) = \left\| \begin{matrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{matrix} \right\|.$$

Similarly, the homothetic group introduces multiplication of a square matrix by a scalar. Complex numbers follow by setting up the correspondence :

$$1 : R(0), \quad i : R(\tfrac{1}{2}\pi).$$

Chapter II introduces permutations and permutation matrices, and the related symmetric group of order 3! is represented by the group of permutations on the coordinate axes OX, OY, OZ. By choosing new coordinates :

$$\xi = x + y + z, \quad \eta = 2x - y - z, \quad \zeta = -2z + y + x,$$

an equivalent matrix representation of this group is obtained in reduced form. This interesting illustration demonstrates how irreducible representations are associated with invariant subspaces.

Chapter III deals with the definition and properties of determinants and with the solution of simultaneous linear equations.

Chapter IV introduces vectors in three-dimensional space and the representation of various transformations of space by means of them. The operations of projection and reflection are interpreted as square matrices operating on column vectors. Finally, the effect on matrix operators due to a change of axes leads to a consideration of latent roots and latent vectors.

This book is unlikely to be much used by British students, although their teachers might find it useful in suggesting novel approaches to familiar topics. From the point of view of the average physicist, for whom presumably the book is intended, it would seem to over-emphasise the axiomatic standpoint, but a pure algebraist would find gaps in the axiomatic foundations, as for example in the definition of the addition of integers (§ 1·1). Groups, although frequently mentioned, appear to be nowhere properly defined. The phrase " logarithmes népériens " on p. 77 provides a smile for Scottish readers.

D. E. R.

The Theory of Mathematical Machines. By F. J. Murray. Pp. 108. \$3. 1947. (King's Crown Press, New York)

The following machines and devices are variously mentioned, sketchily described or analysed : the abacus, key-board and relay adding machines, the Leibnitz wheel, Napier's bones, the " Millionaire " and " Euklid " digital computing machines, the Hollerith punched-card machine, string and pulley devices for addition and subtraction, lever and linkage systems, the differential gear, the Wheatstone bridge, potentiometer schemes for addition and subtraction, the coiled-tape, spiral-gear and other means for squaring (to give multiplication on the basis of the formula $(a+b)^2 - (a-b)^2 = 4ab$), various planimeters, various harmonic analysers, the watt-hour meter as an integrator, the Kelvin ball and disc integrator, the speedometer, a condenser-resistance circuit network to give the integral and the derivative of the e.m.f., the gyroscope, servo-motor controls, mechanical torque amplifiers, filters and electronic amplifier networks, the selsyn and differential selsyn system, the Bush differential analyser, the Mallock machine, the cinema integraph and the new Automatic Sequence Controlled Calculator at Harvard University

The discussion for the most part is very incomplete, but by way of compensation reference is made to the book by McColl on *Servo Mechanisms*, to the book by Bode on *Network Analysis and Feedback Amplifier Design*, to the *R.C.A. Receiving Tube Manual*, to numerous other commercial hand-books and to numerous articles in the journals of the engineering societies. The book is the outgrowth of a course of lectures given recently in the Department of Mathematics of Columbia University.

Although he does not say so, the author's primary concern obviously is with an exploration of the possibilities of realising certain new means of computation in a class of problems. The first sign of this secret purpose occurs on p. 66, where it is recalled that while there is no theoretical difficulty in obtaining a true harmonic representation of any function which is periodic and continuous with continuous derivatives on $-\pi \leqslant x \leqslant \pi$, a Fourier series representation of a non-periodic function on that interval may not be uniformly convergent and term-by-term differentiation may not be possible. The Fourier series $y = 2\Sigma_{i=1}^{\infty}(-1)^{n+1}(\sin nx)/n$ for the function $f(x) = x$ is a case of this kind. But in dynamical and electric network problems, a series approximation of a function f is often desired which approximates the function and whose first two term-by-term derivatives simultaneously represent f' and f'', respectively, equally well.

The author considers the approximation to the function f by a trigonometric sum s_n in which the coefficients are determined so as to minimise the error integral

$$\int_{-\pi}^{\pi}(|f - s_n|^2 + |f' - s_n'|^2 + |f'' - s_n''|^2)\,dx.$$

If the numerical value of this integral can be made as small as desired short of zero by a suitable choice of n, then the approximate representation s_n of f may be of greater utility than the Fourier series representation of f, if it exists, particularly if some sort of harmonic analyser can be devised for the computation of the coefficients.

The orthogonal series representation of a function is regarded as a vector in an infinite dimensional function space in which the functions 1, $\sin x$, $\sin 2x, \ldots, \cos x, \cos 2x, \ldots$, correspond to reference vectors or coordinate axes. In this space the inner product between two vectors f and g is defined to be the number

$$\int_{-\pi}^{\pi}(fg + f'g' + f''g'')\,dx$$

so that the lengths of the vectors **1** and $\sin nx$ are $\sqrt{(2\pi)}$ and $\sqrt{\{(1+n^2+n^4)\pi\}}$ respectively, which last is also the length of the vector $\cos nx$. Unfortunately, this reference system of vectors is not complete, for there exist vectors in this "space" orthogonal to all of them. Let f be such a vector and let ϕ stand for any of the functions 1, $\sin nx$, $\cos nx$. Then

$$0 = \int_{-\pi}^{\pi}(f\phi + f'\phi' + f''\phi'')\,dx = \Big[(f' - f''')\phi + f''\phi'\Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(f - f'' + f^{\text{IV}})\,\phi\,dx,$$

from which it is found, with the substitution $\phi = \sin nx$, that $f''(\pi) = f''(-\pi)$, and with $\phi = \cos nx$, that $f'(\pi) - f'''(\pi) = f'(-\pi) - f'''(-\pi)$, and hence

$$0 = \int_{-\pi}^{\pi}(f - f'' + f^{\text{IV}})\phi\,dx$$

for all ϕ. Hence the three conditions :

$$f - f'' + f^{\text{IV}} = 0, \qquad \Big[f''\Big]_{-\pi}^{\pi} = 0, \qquad \Big[f' - f'''\Big]_{-\pi}^{\pi} = 0,$$

the first of which follows from the equation preceding it by a well-known theorem on Fourier series, are necessary if f is to be orthogonal to all the ϕ's, and they are clearly sufficient ; one finds

$$f = A \exp(x\sqrt{\tfrac{3}{2}}) \sin(\tfrac{1}{2}x + \gamma_1) + B \exp(-x\sqrt{\tfrac{3}{2}}) \sin(\tfrac{1}{2}x + \gamma_2).$$

Accordingly, the reference frame becomes complete by the addition of the two vectors, namely :

$$f_1 = \exp(x\sqrt{\tfrac{3}{2}}) \sin(\tfrac{1}{2}x - \tfrac{5}{6}\pi) + \exp(-x\sqrt{\tfrac{3}{2}}) \sin(\tfrac{1}{2}x + \tfrac{5}{6}\pi),$$

$$f_2 = \exp(x\sqrt{\tfrac{3}{2}}) \cos(\tfrac{1}{2}x - \tfrac{5}{6}\pi) + \exp(-x\sqrt{\tfrac{3}{2}}) \cos(\tfrac{1}{2}x + \tfrac{5}{6}\pi),$$

which are orthogonal to each other and have lengths C and $C\sqrt{3}$ respectively, with $C = 2\sinh(\pi\sqrt{3})$. An arbitrary function f can be represented on the interval $-\pi \leqslant x \leqslant \pi$ by the series

$$\sigma = \Sigma_{n=1}^{\infty} a_n \sin nx + \Sigma_{n=1}^{\infty} b_n \cos nx + c_1 f_1 + c_2 f_2,$$

in which the coefficients are given by

$$a_n, b_n = \frac{1}{\pi(1+n^2+n^4)} \int_{-\pi}^{\pi} (f\phi + f'\phi' + f''\phi'')\,dx, \quad b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx,$$

$$c_1 = \frac{1}{C} \int_{-\pi}^{\pi} (ff_1 + f'f_1' + f''f_1'')\,dx, \quad c_2 = \frac{1}{C\sqrt{3}} \int_{-\pi}^{\pi} (ff_2 + f'f_2' + f''f_2'')\,dx.$$

This representation of an arbitrary function f has the property that the representation and the term by term first and second derivatives converge in the mean to f, f' and f'', respectively ; Hardy and Rogosinski, in their Cambridge Tract No. 38, Theorem 16, p. 17, show that if a space of this kind is complete it is also closed.

The functions f_1 and f_2 are linear combinations of the linearly independent solutions of the equation $y'' \pm \sqrt{3}y' + y = 0$, which solutions can be realised in two electrical networks in which the inductance and capacity are each unity in both cases and the resistance is $\pm\sqrt{3}$, respectively. Thus there are means available to supply electrically all the functions f_1, f_2, $\sin nx$, $\cos nx$ with respect to which the coefficients are to be computed. The possibility of constructing an electrical analyser of the harmonic analyser class which will give the approximate representations of the kind described becomes apparent : the machine would minimise the error integral by continuous or successive adjustment of the coefficients but with

$$\sigma_n = c_1 f_1 + c_2 f_2 + \Sigma_{m=1}^{n} a_m \sin mx + \Sigma_{m=1}^{n} b_m \cos mx$$

replacing s_n.

A simpler problem of adjustment is discussed on p. 91, where the possibility of machine solution of the equation $Ax = b$, A an $n \times n$ matrix, x, b vectors, is considered. Let u be a tentative solution of the equation, let $e = Au - b$, let W be a weighting matrix, and let μ be the square of the weighted discrepancy vector We. Then

$$\mu = \cdot eW^2e{:} = (uA_T - b)\,W^2\,(Au - b),$$

in which A_T is the transposed form of A and $\cdot e$ and $e{:}$ are the line and column arrangements of the vector e. Define $\cdot\text{grad}\ \mu = \left(\frac{\partial\mu}{\partial u_1}, \frac{\partial\mu}{\partial u_2}, \ldots, \frac{\partial\mu}{\partial u_n}\right)$, μ any scalar, and suppose u depends on t. Then $d\mu/dt = \cdot\text{grad}\ \mu\ du{:}/dt$. Obviously, if $du = -\text{grad}\ \mu\lambda(t)\,dt$, $\lambda(t)$ positive, then any change du computed by this equation will diminish μ and hence give a vector $u + du$ nearer to the solution if du is not too great ; one uses the formula

$$\cdot\text{grad}\ \mu = 2(uA_T - b)\,W^2A, \quad \text{or} \quad \text{grad}{:}\ \mu = 2A_TW^2(Au - b).$$

There is, of course, the question of what to take for $\lambda(t)dt$ in ordinary procedure by successive adjustments. If too small a numerical value is taken, then the improvement in μ is not great enough, and if too great a numerical value is taken the process overshoots itself and μ is increased instead of being decreased : a smaller $\lambda(t)dt$ is substituted for the unsuccessful value and the resulting effect on μ tested. That the process will converge exponentially in the case of continuous adjustment is proved as follows. Since $A_T W^2 A$ is symmetric, a matrix T, det $T \neq 0$, exists such that $TA_T W^2 T^{-1} = N$ is a diagonal matrix. With the substitution of variable $u = vT$ we have

$$du = dv\,T = -2(vTA_T - b)\,W^2 A\lambda(t)\,dt,$$

whence

$$dv = -2(vN - bW^2AT^{-1})\,\lambda(t)\,dt,$$

so that

$$dv_i = -2(v_iN_i - a_i)\lambda(t)dt,$$

and

$$v_iN_i - a_i = \exp\left(-2N_i \int_{t_0}^{t_1} \lambda\, dt\right),$$

where a_i is the ith element of the vector bAW^2T^{-1}, N_i the ith diagonal element of the matrix N, and v_i the ith element of the vector v. If A is the kernel of a definite form, the elements N_i have the same sign for all i, so that, ultimately,

$$v_iN_i - a_i = 0,$$

whence

$$vTA_TW^2T^{-1} - bAW^2T^{-1} = 0,$$

and so

$$Au_\infty - b = 0.$$

In the case of discrete adjustment, the inevitable over-shooting mentioned always results in a considerable amount of wasted effort, and this no doubt accounts for the failure of the method to be mentioned in recent discussions and bibliographies on numerical methods of solution of equations (*e.g.* Hotelling, *Annals Math. Statistics*, 14 (1943), p. 440). However, if a machine could be constructed to make a very large number of small adjustments in u per unit time, or to make them continuously (which is impossible), then the danger of over-shooting will not arise until the end of the process when the defects of the machine, back-lash and the like, become decisive. This is what Murray proposes, and the main features of an arrangement of machine elements for realising the necessary mathematical operations are diagrammised.

It is claimed that a device for making adjustments in tentative solutions of non-linear equations in the direction of the negative gradient of the squared discrepancy is possible : one writes any system of such equations in the form $f_i(x_1, x_2, \dots, x_n) = 0$, $i = 1, 2, \dots, n$, and substitutes e_i on the right-hand side when the x's are replaced by the tentative solution. Adjustment of the latter is made in the direction of the negative gradient of $\mu = e_ie_i$, but how the machine is to ascertain the gradient is not explained.

The problem of approximate solution of the second order differential equation $F(x, y, y', y'') = 0$ on the interval $a \leqslant x \leqslant b$ with boundary condition $G(y(a), y'(a)) = 0$, $H(y(b), y'(b)) = 0$ is also considered. The error function taken is $\mu = \int_a^b F^2\,dx + G^2 + H^2$, and it is proposed to substitute σ_N, defined above, as the approximate solution. Reference is made to a vacuum-tube feed-back element in the possible computing machine. The same kind of approximate solution to a general second order partial differential equation with a condition on a space-curve boundary is also discussed.

There is an excellent discussion of the Gauss-Seidel process, including a description of novel electrical means which realise the process. Evidently the last machine calculator has not been invented, and this is not the author's last word on the subject.

Any inventor interested in the problem of the numerical solution of equations of the kind mentioned in this review will find this little book, with all its shortcomings in completeness and exposition, informative both mathematically and technologically. W. H. I.

Modern Electrical Engineering Mathematics. By S. A. STIGANT. Pp. 369. 31s. 6d. 1947. (Hutchinson)

In his preface the author states that this book is intended to present, in simple introductory form, outlines of some of the progress which has been made in the application of the results of pure mathematical research to the solution of problems arising in electrical theory and practice.

The first few chapters deal with the properties of r, j, and $e^{j\theta}$ from, as is usual in technological studies, the standpoint of vector operators. Determinants are explained and their use in solving simultaneous equations illustrated.

The greater part of the work is devoted to the development of the matrix, dyadic, and tensor calculuses and their application to the solution of linear network problems in power engineering. The book concludes with chapters on the theory of symmetrical components, the Heaviside operational calculus with applications to transient phenomena, dimensional analysis, the per-unit method, and the relation between the two latter and the tensor calculus.

It will be seen that the author has attempted to cover a great deal of ground in this one volume, and it might well have been better if several of the topics treated had been reserved for a later volume. Fuller explanations could then have been given of such matters as the concept of a reference frame as understood in network theory, the definition of a tensor, covariance and contravariance, and similar subjects of fundamental importance, ideas which will probably be strange to most of the readers for whom this book is intended. As it is, the more theoretical parts of the book are very condensed, and the author's rather loose style does not help towards a clear understanding. It is, of course, always difficult when addressing those whose chief interest is not mathematics to know exactly where to draw the line between a precision in explanation which will lead towards a clearer understanding and that which will be dismissed by the reader as pedantry and labouring of the point.

It is, perhaps, unfortunate that it was decided to treat transient phenomena by the method of the Heaviside calculus rather than by that of the Laplace transform, which, if one may judge from the technical journals, is rapidly gaining favour on account of its wider applicability and greater clarity.

Throughout the book there are many applications of the mathematical techniques treated to the solution of practical problems, generally in three-phase networks, and several such problems are solved most instructively by two or more methods. In short, one may say that the author has succeeded in writing a book for the electrical engineer who wishes to learn the use of the more advanced mathematical techniques and who is prepared not to understand fully the underlying theory at a first or second reading. The many references in the text and the bibliographies appended to most chapters will tell him where to seek further information.

The make-up of the book unfortunately reflects the current difficulties which confront all authors and publishers. The lay-out is cramped, the paper of poor quality, and the diagrams and lettering are not always as clear as is desirable. L. M. H.

The Escalator Method in Engineering Vibration Problems. By JOSEPH MORRIS. Pp. xvi, 270. 21s. 1947. (Chapman and Hall)

The author's work over many years as consultant in the Structural and Mechanical Engineering department of the Royal Aircraft Establishment has presented him with (among others) a number of complicated examples of the vibration problems which afflict more and more all branches of engineering, but which are especially acute in aircraft, where mass-stiffness ratios are reduced to a minimum and where no comparatively rigid or massive supports exist.

The central problem is the determination of the roots of the Lagrangian frequency equation, and of the associated modes. The frequency equation is usually presented as a determinant which expresses the consistency of a set of simultaneous linear homogeneous algebraic equations in the coordinates, derived from the equations of motion upon the hypothesis of the existence of normal modes. Even when considerably idealised, and with continuous systems crudely " lumped ", the number of equations and the order of the determinant (*e.g.* for a six-cylinder engine driving a four-bladed propeller whose flexural vibrations must be considered) is such as to daunt the hardiest computer from a frontal attack.

The author's contribution is what he terms the " escalator " method. It consists essentially in a gradual approach, starting with systems of low order and increasing the order by unit stages. The equations determining the frequencies and modes of a system of order $(n+1)$ are simply expressed in terms of the frequencies and modes of a system of order n and the additional (" coupling ") terms.

The scope of the book is, however, much wider than the title implies, for it forms a more or less connected and complete account of the problems encountered and solved by the author in an extended research career. Early chapters deal with statical elastic systems (including Castigliano's principle, the Hardy Cross method, and the Rayleigh reciprocal theorems) expressed consistently in the language of influence coefficients. Coming to vibration problems, an account of other methods (Rayleigh's principle, Dunkerley's rule, the Duncan and Collar iteration process) is first given, and then the " escalator " method is introduced. Complete numerical results for a sixth-order system, including full details for one stage, are given. Starting with a second-order system, needing the solution of a quadratic equation, we proceed through systems of order 3, 4, 5, to that of order 6 whose solution is desired. It is also shown how the escalator can function " in reverse ", *i.e.* making the equations of a system of order $(n-1)$ depend on the solution of those of order n. The method is also adapted to solve the simpler problem of a set of simultaneous linear algebraic equations.

The second half of the book seems to have the aim of formulating the many vibration problems occurring in aircraft, and of breaking them down into the " coupling " of simpler systems. For example, torsional vibrations of the engine crankshaft, and flexural vibrations of the propeller blades, are first considered separately, and the complete system is attacked by calculating frequencies for the two parts in terms of an assumed position of a node on the shaft as a parameter ; the required frequencies are obtained from the intersections of the graphs of the two sets of frequencies so obtained. Vibrations of the engine-fuselage system, and of the fuselage-wing system, are considered, and the frequencies (overtones) which survive in the case of a multi-cylinder engine are discussed. Finally, some attention is paid to the so-called " pendulum vibration damper "—which, as the author remarks, would be more accurately described as a " dynamic de-tuner ".

In appraising this book, one must perforce discriminate sharply between matter and manner. As regards the escalator method itself, it is clear that, whatever one's reaction to the numerical tables (*e.g.* pp. 119–122) and to the amount of arithmetic implied, may be, the method does present a systematic approach by easy stages to a complex problem; it has been successfully applied by its inventor to systems of very high order; it has been used in preference to other methods with which the author was well acquainted, and which he describes quite fairly. To one's doubts as to whether the labour of solving *accurately* (as is imperative) all the systems of lower order is economic, the author might reply, "I've tried all the others, but now I 'escalate' every time!" Moreover, the method yields equally readily *all* the frequencies and modes of a system, and the results for stage n make it easy to obtain *good* first approximations to those of stage $(n+1)$. But, in addition to its declared purpose, the book is of value as a review of a variety of statical and dynamical problems, and of methods which have been invented for their numerical solution. The historical introductions to many chapters are good to have, and reflect the width of the author's reading.

As regards presentation, the book is that of an engineer and a researcher, interested in *results*, rather than that of a mathematician and a teacher, interested in *methods*. The clear, orderly, and detailed presentation of the numerical examples contrasts with that of the more theoretical sections. The accounts of principles in the early chapters are (perhaps intentionally) sketchy and incomplete; occasionally they are inaccurate. Every now and then steps in the argument are not well sign-posted; symbols are not always judiciously chosen; diagrams are not very clear; one gets an impression of considerable repetition, and chapters end inconclusively.

But in spite of these defects the book is both informative and suggestive. One feels that there may well be in the background some generalising principle awaiting discovery—and perhaps exploitation. Possibly a more succinct mathematical formulation would enable one to discern the "wood" from the "trees". Finally, is there any hope that we may be able to mount (or descend) the escalator two or more steps at a time? W. G. B.

Achievements in Optics. By A. Bouwers. Pp. 135. 12s. 1946. Monographs on the progress of research in Holland, 1. (Elsevier Publishing Company, New York and Amsterdam; Cleaver-Hume, Ltd., 42a South Audley Street, London, W. 1)

In 1929 T. J. I'A. Bromwich wrote: "The colossal industry of moving pictures depends quite as much on the less picturesque work of the patient labour of interested mathematicians as on that of the film stars at Hollywood." The battle in the field of design has been mainly over the control of the five Von Seidel and higher aberrations. The first modern photographic anastigmat is said to have taken five years to design. Fraunhofer also constructed a system of four surfaces. There are probably fifty operations for each ray which meets the axis and more than 200 for each skew ray. No wonder the field developed more writers than readers. Nevertheless, no one has seen the rings of Saturn for the first time without feeling a sense of admiration for the optical artistry. The volume before us is an attempt to bridge the gap between the gruelling detail of modern technique and a popular account for the general scientific reader.

The work is divided into four sections: I. New optical systems; II. New optical instruments; III. Geometrical optics; IV. Physical optics. The first section deals with the comparatively recent Schmidt systems, which have been so well described in this country by Burch and Linfoot. B. Schmidt

was responsible for the novel idea of introducing a non-spherical element at the centre of curvature of a spherical mirror so as to correct the aberrations. New optical mirror systems are here described, where the method is extended to include the combinations of a spherical mirror and a suitable system of lenses. The remarkable concentric mirror system is again improved by a corrected concentric system in which an aspherical corrector is placed at the centre of curvature of the concentric mirror system with small residual spherical aberration. Like the Schmidt corrector, it has rotational symmetry and its shape is such as to correct the residual spherical aberration. Since the stop is placed at the centre, there are no other aberrations apart from small higher order aberrations. The Rayleigh tolerances are fulfilled.

Among the new optical instruments, there is a simple description of a new microscope embodying mirrors and correctors; a new telescope in which the total length is reduced to one-third of the focal length, and new monocular and binocular field glasses. There is an excellent photograph of a face illuminated by a match, taken by an improved Schmidt camera. A new spectral camera for astronomical use is shown, and the special properties of the concentric mirror system open up new possibilities in wide-angle photography. The new concentric mirror system seems also equally suitable for television projection. On p. 2, formula (1), the last term should be $3h^4/128F^3$.

The mathematical treatment of the propagation of light is usually based on two theories: (i) geometrical optics, (ii) physical optics (wave theory). These seem different and can be developed independently. Actually they are intimately connected and it is now generally agreed that both points of view are necessary even to a designer. If we start with Clerk Maxwell's electromagnetic theory of light, we can take the wave theory as the general theory, and geometrical optics becomes that part of the wave theory which describes the propagation of light signals in sudden discontinuities.

In the section on geometrical optics there is an interesting geometrical construction of the meridian and sagittal image points. The researches on aberrations are based on the work of T. Smith (without his matrices), starting from the basic ideas of Sir W. R. Hamilton and the Eikonal of Bruns. There are also interesting extracts of papers dealing with geometrical aberrations suggested by the wave treatment of physical optics, the luminosity of optical systems and the light distribution in image points and lines. The illustrations are excellent, and there is a feast of ideas here for the mathematician.

Under the heading of physical optics, the diffraction theory of aberrations is dealt with. The author does not start with Clerk Maxwell's equations, but with Kirchhoff's principle obtained from Green's theorem as a consequence of the wave equation satisfied by the light amplitude. The usual formulae are developed by means of Bessel functions from the intensities of the Airy star disc to the diffraction pattern associated with an arbitrary single aberration. Two examples are given, and here is a field which mathematicians could work with great profit, although the theory is in danger of outstripping the practice. Finally, there is a simple readable account of an extension of Rayleigh and Abbe methods due to Professor F. Zernicke dealing with phase contrast for microscopic observation of transparent objects. The twofold nature of the diffraction is emphasised. Not only does every point on the aperture of the objective contribute to the vibration in a selected point of the image, but also every point in the object as well. By the phase-contrast method transparent details of the object which differ in thickness or refractive index appear as differences of intensity in the image. Sensitivity can thereby be obtained by use of an absorbing phase strip placed and adjusted in the microscope.

The publishers are to be congratuated on this production, which is lavishly

illustrated and printed on good paper. It is one of a series of monographs in English on the progress of research in Holland during the war. Part of the work was carried out secretly under the burden of oppression and starvation prevalent during the occupation. The average reader will find much to interest him in the range of subjects discussed, and this is a fitting volume from the country which was the home of spectacles and which cradled Snellius and Huyghens.

A. Buxton.

The Teaching of Arithmetic and Elementary Mathematics. 2nd edition. Pp. vii, 255. 7s. 6d. 1946. **The Teaching of Science.** 3rd edition. Pp. 194. 7s. 6d. 1947. By W. L. Sumner. (Blackwell, Oxford)

The first of these books is the second edition of a book originally published in 1938. In his preface the author states that he has taken the opportunity to make a number of additions for the sake of teachers in the secondary modern schools, and the result is a volume of nearly 250 pages which should be read by all teachers of mathematics.

The usual subjects are treated in turn from the very beginning, but the book does not deal with post-School Certificate work. Indeed, it scarcely reaches School Certificate standard as, although there are continual allusions to secondary schools, it is not always clear whether the author is referring to grammar or to modern schools. His terminology is sometimes confusing. He speaks of central schools, senior girls' schools, and often enough, when he talks of secondary schools, he really means the modern schools. It is obvious, however, that in general he is alluding to the latter, as he definitely states on p. 46 that he does not mean to say much about work in grammar schools, but he clearly hopes that teachers in the latter will profit by his remarks, as he is somewhat critical of the academic character of their teaching. Further, he claims that the additional year or two at the secondary modern school gives an opportunity of recasting their syllabus, and methods of teaching. Such an object is wholly admirable, and the book is both challenging and ambitious.

The earlier chapters deal with Arithmetic and Geometry in the primary schools. Mathematics in the secondary comprise, in turn, Arithmetic, Algebra, Geometry and Trigonometry ; and then in the last fifty pages there are some pleasant notes on the Calculus and various bye-ways of Mathematics. Much of the course is naturally common to all types of schools.

In the Arithmetic section the author quotes Cajori to the effect that " the miraculous powers of modern calculation are due to three inventions ; the Arabic Notation, Decimal Fractions and Logarithms ", and he develops his suggestions on this basis. His method of dealing with percentage will not be accepted by all, but few will disagree with his criticism of unreality in arithmetical proportion. " A cricketer made 10 runs in half an hour. How many would he make in an hour and a half? " Indeed, it is interesting to find how he develops similarity from ordinary unitary method. There is also some good advice on money questions, and the various methods of approach. There is an excellent note on decimalisation to three places at sight, a most valuable accomplishment, which should surely be more widely used than it is. The author does not fail to point out the limitations of the method.

Then comes Algebra. We welcome the pages on directed numbers, and his few practical examples, but it is a pity that there is a misprint in the note on p. 114 where he is illustrating the fact that large indices produce numbers beyond human comprehension. There is, further, the usual discussion on graphs and logarithms, where modern methods are presented on sound lines.

He is careful not to stress deductive geometry; indeed, he clearly states that Intuition and Induction (Experiment) must take up most of the time available at a secondary modern school, and that there will be little time for deduction. But he includes a considerable amount of Mensuration of the highest value, though some of it may be rather too hard for children at the modern school. Can they be expected to do much on the lines of Orthographic and Radial Projection? It is, however, good to note that he gives importance to π and encourages the teacher to pass on some of its history to the pupil. Trigonometry naturally follows, though we are not quite happy about his introduction of non-acute angles. There is an extension to surveying, and problems concerned with latitude and longitude which are reminiscent of the work done by Mr. P. F. Burns, explained in his recent talk to the Mathematical Association.

The last fifty pages are more general. " Going Ahead " is the main title. Presumably little of this will be taught at present in the secondary modern school. Like others, he is all for an early introduction of the calculus. His chapter on " Civic and Rural Arithmetic " is worthy of the attention of all teachers.

The author is certainly ambitious in his outlook on the work to be expected in secondary modern schools; but an ideal should be there, and teachers can at all events aim towards it. The book is most stimulating, and it should be read and re-read by all teachers. The Mathematical Association has done a great work during the past forty years, and there is a well-deserved tribute to its many " excellent publications " in the introduction. Most of these, however, have had the grammar schools in mind; now, with a higher school-leaving age there are large numbers of children whose syllabuses must be re-modelled; the work must be concrete, and above all stimulating; there is little limit to the possibilities, and we welcome the publication of this new edition, which is a bold and excellent attempt to direct teachers along a satisfying course. The book deserves a wide sale among all concerned with teaching Mathematics to the young.

The second book is a companion volume, and is the third edition of a book published in 1936. It was originally written to give " a short survey of science teaching "; in his latest edition the author has mainly in mind the needs of the new secondary modern schools, though much of what he says obviously applies to all schools where science is taught.

It is an advantage that the same author should give advice on both mathematical and science teaching. Some forty years ago many attempts were made by mathematical and science masters to correlate their work. These were fruitful of result, and although less is heard of it to-day, the need is possibly as great as ever. Indeed, the author pleads for a system of mutual help. Science, too, is younger as a school subject, and it is valuable that a man of Mr. Sumner's knowledge and experience should give some account of what may be attempted in the new secondary modern schools.

The author is ambitious, and rightly ambitious; he is conscious of the comparatively short school life of most children. He is entirely in favour of a system of General Science, as it is popularly called. He hopes that between the ages of 11 and 15, elementary Chemistry, Physics and Biology may be taught, including in the latter some instruction in elementary physiology and hygiene. He urges strongly the claims of " the four books in the series *An Introduction to Science* by Andrade and Huxley ", and states that " the reading of such works will result in the stimulation of interest, and the appreciation of the relation of science to life in all its phases—which are points of paramount importance ". Like all good teachers, he is opposed to expensive apparatus; " it is possible to do some excellent science teaching without a

laboratory, indeed even without gas or electricity supplies." Or again, " a considerable amount of apparatus for biological teaching may be made in the school workshops ". Few will disagree.

The author first defends the claims of Science as a school subject, and emphasises the moral aspect; the utilitarian side is so easily grasped that the sense of responsibility is left behind. " The guardianship of its uses must be a duty for the future."

Then comes the consideration of general methods with a recognition of the comparatively short time available. We have rarely read a better account of what the demonstration lesson should be, and then comes a suggested syllabus for both A and C children; it is refreshing to find how continually he appeals to the facts of everyday life.

The eighth chapter on Astronomy and Meteorology strikes newer ground, but surely he can be hardly right in demanding a " partial explanation of Einstein's theory of Relativity ", which, he states, is " almost a necessity to the Senior and more intelligent children ". A most laudable ambition, but hardly possible for most teachers! This is followed by forty pages on Biology and Hygiene, well and concisely treated. He concludes with an account of aids to science teaching by means of posters, optical instruments and visits to museums amd factories; nor is the science library or the equipment of a laboratory forgotten.

The book is shorter than its companion volume, but is very stimulating. Every teacher will profit by reading it; educational advance is slow; the new science teachers must be trained, but it is of high value that there should be produced for them a book proposing so admirable a course at which to aim.

W. F. Bushell.

General Mathematics. By C. V. Durell. Vol. 3. Pp. xxxii, 332, xxxvii. With answers, 6s.; without answers, 5s. 6d. 1947. (Bell)

This is the third volume of the series of which the first two volumes were reviewed in Vol. XXX, No. 292. December, 1946. It consists largely, as the other volumes, of a new arrangement of Mr. Durell's textbooks, but there is here more new material, and the Trigonometry and Geometry do look more like the homogeneous course aimed at in the series.

Full use of tables is clearly intended in this volume, and it starts with tables of logarithms and both logarithmic and natural sines, cosines, and tangents. The chapters for the first half are on miscellaneous arithmetic, use of logarithms, areas of parallelograms, etc., quadratic equations, Pythagoras' theorem, trigonometry in three dimensions, simple interest, and algebraic fractions.

In the chapter on miscellaneous arithmetic there is some interesting material. Modern methods of finding averages lead to a treatment of weighted averages —it is good to see the correct terminology introduced—and of average heights of curves, including Simpson's rule. There is then an instructive section on gas and electric meters and on electrical energy, which will often interest and be useful to parents as well as children. The chapter on Trigonometry in three dimensions includes a section on latitude and longitude with methods of finding them, using pole star and time. Simple interest is enlivened by a section on an instalment system. This shows that the book has an appeal to pupils through matters of current interest.

This volume is ambitious, and is only likely to be completed in the year at age 13–14 by the best boys. Beyond the subjects mentioned there is a section on the solution of triangles by sine and cosine rule, and one on simultaneous equations with three unknowns, and with two unknowns one equation being

quadratic. In fact with a few omissions it completes a School Certificate course.

The geometry is well interspersed with calculations, many involving trigonometry, but, perhaps in consequence, there are very few sections on constructions, a side of geometry which is appreciated by the slower pupil. In the solution of triangles the sine and cosine rule are proved and used, both in riders as well as calculations, but later in the chapter the extensions of Pythagoras are still included, perhaps unnecessarily. The geometry does still appear rather formidable, in its formal lay-out and in the amount included, for a pupil of such immature years as this book is intended for.

There is a set of revision examples in the middle of the book and at the end there are 48 tests in computation and 90 revision papers, some of each being on the preceding volume. K. S. S.

Examples in Engineering Mathematics, Book II. By I. R. VESSELO and S. H. GLENISTER. Pp. 87. 3s. 6d. 1947. (Harrap)

This is published as a set of Examples for the 2nd Year of National Certificate and is to be followed later by a Book III for the 3rd Year Course.

The authors have already established themselves with other publications. This work opens with a set of Examples as a revision exercise on the work of the First Year. Then follow groups of Algebra, Geometry, Trigonometry and Calculus.

The Algebra sets treat simultaneous and quadratic equations, arithmetic and geometric progressions; the Geometry covers the properties of the circle, similarity, the cone, the pyramid and the sphere, including frustra and zones, the area of irregular figures and projections; the Trigonometry section has questions on the graphs of trigonometrical functions, the general angle, compound angles, solution of triangles, area of triangles and easy co-ordinate work leading to the determination of laws. Examples on gradients precede questions on simple differentiation and integration. The book concludes with two comprehensive Test Papers.

Tables of logarithms, sines, cosines, tangents, form a handy and useful addition. The whole book is clearly printed and well produced and should prove very useful to the keen student. E. J. A.

Shorter School Arithmetic. By G. H. R. NEWTH. Pp. 248. 1947. 4s. 3d.; with answers, 4s. 9d. (University Tutorial Press)

Shorter School Arithmetic is based on the *Tutorial Arithmetic* by Workman, and is intended to meet the requirements of those teachers who prefer a condensed course, particularly for those who require a revision course for the School Certificate year. The book is in two parts. Part I, which occupies only 67 pages, consists of a summary of the subject-matter dealt with in Part II and contains no examples. Most of the topics are dealt with very briefly, and it seems that this part might as well have been omitted altogether or confined to essential tables and formulae.

In Part II explanations and methods are omitted, it being left to the teachers to use the methods they prefer. The saving of space, as a result of this plan, allows for full sets of examples which would satisfy the needs of those teachers who desire a longer course than is implied by the title of the book.

At the end there is a collection of 185 questions taken from School Certificate Examinations. S. I.

VISUAL MATHEMATICS

PORTRAITS.

Portraits of Eminent Mathematicians. Portfolio I. With brief biographical sketches by D. E. SMITH. New de luxe edition. \$5.00. 1946. (*Scripta Mathematica-Yeshiva University*, 186th Street and Amsterdam Avenue, New York)

The portfolio contains a reproduction of a mosaic, depicting the death of Archimedes, together with portraits of Copernicus, Viète, Galileo, Napier, Descartes, Newton, Leibniz, Lagrange, Gauss, Lobachevsky and Sylvester. The plates are clearly and beautifully engraved, and are very suitable for classroom display. Each is accompanied by a short biographical note, set out for display with the photograph. Facsimiles of handwriting and engravings of the title pages of famous books are also included.

Teachers will welcome the new edition of this excellent collection, which brings these eminent mathematicians to life. They should find the portraits and biographies an adornment to the classroom and a stimulus to the pupils. It is to be hoped that the death of Professor Smith will not destroy the promise of further selections from his personal collection of portraits. I. R. V.

FILM STRIPS.

This new medium, a modern development of the lantern slide, much used in the Forces as a means of applying mass production methods to instruction, is rapidly gaining ground in the schools, as an aid to teaching. Each strip consists of between twenty and thirty pictures, and is accompanied by a set of teaching notes. Although the range of subject on which film strip is available is wide, only the bolder spirits amongst producing companies have so far entered the realms of mathematics. Common Ground Ltd. have produced three strips which are noticed below, whilst several are promised by Educational Publicity Ltd. in January. Of others we have no information.

Laws of Growth. Part I. Building Linear and Polynomial Laws. By R. A. FAIRTHORNE, B.Sc. (CGB 182.) 25 pictures. 9s. 6d. Teaching notes, pp. 21. 3s. 1947. (Common Ground Ltd.)

This strip deals with laws built up by the addition of equal increments. Beginning with the simplest additive scale, the ruler, and the cactus, which adds one branch at each stage, it goes on to the stair, a pictorial representation of the arithmetic series. Using the terms of this series as increments for the next, the identities for $\Sigma(n^2)$ and $\Sigma(n^3)$ are illustrated by series of figurate numbers, represented by patterns of blocks. The next pictures show the process of interpolation from two, three and four given values. This is followed by diagrams showing how an error in table-making is traced to its source, and pictures of the National Accounting Machine illustrating its use in the calculation of tables.

Mr. Fairthorne, whose pioneer work on mathematical films is well known, has put together the material for an interesting line of study. The pictures are well chosen and well drawn, although the figurate number patterns are somewhat difficult to follow. The photography is excellent. Although the subject-matter is outside the range of normal school work, it has many links with it and provides much useful material. The teaching notes are stimulating but perhaps too severe.

Laws of Growth. Part 2. The Exponential Law and its Applications. By R. A. FAIRTHORNE, B.Sc. (CGB 183.) 27 pictures. 9s. 6d. Teaching notes, pp. 21. 3s. 1947. (Common Ground Ltd.)

Growth in accordance with a constant ratio is illustrated by the slide rule, the piano keyboard, optical filters and fission of cells. Interesting geometrical applications introduce golden section, the equiangular spiral and growth of regular and irregular figures (natural shells) by gnomon. Other examples are obtained by rolling an expanding square, hexagon and circle. The remaining pictures deal with the circular slide rule, the wheel and disc integrator, and the representation of negative growth ratios.

A fascinating set of pictures has been provided, to develop an interesting theme. The artist's interpretation of the mathematical ideas is praiseworthy; the photography is of the usual high order. The subject-matter will invigorate a sixth form; but the strip should never be shown unless both teacher and class are fully prepared. The teaching notes are good, but necessarily brief; they should be supplemented by reading d'Arcy Thompson, *On Growth and Form*.

Introduction to Graphs. By A. C. BARRETT. (CGB 263.) 32 pictures. 9s. 6d. Teaching notes, pp. 14. 3s. 1947. (Common Ground Ltd.)

At the other end of the scale we have a series of pictures to introduce children under ten to the graph. Opening with a modernistic group of featureless children, which turns out to be the seven ages of Tom Jones, it arranges them in sequence and draws a graph of height against age. Other topics of the usual type, such as weight, height of a burning candle, speed, temperature, attendance, position in class, and road casualties are similarly treated. Such captions as " There is always a reason for a sudden kink in a graph " and " Sometimes graphs may help you to prophesy what will happen next ", are inserted to direct the young mind to the problem of interpretation. In several cases the axes are marked for projection on a blackboard, so that the teacher may draw his own graph.

The pictures have been well drawn and well photographed. The explanation is adequate for this age group and will interest the class. Yet it is doubtful whether the direct approach to the graph of functional relation is justified without any reference to the commoner visual methods of representing statistics, such as the histogram and the isotype. The choice of unit is not good; it would appear from one graph that the top position in class is nought; the comparison of road casualties at different ages is made without any reference to age-group totals. The teaching notes are scanty, but perhaps there is little more to be said at this level.

There is much here that will be of help to the teacher of young children. The experiment of a visual approach to this topic is worth while, and Mr. Barrett and his colleagues are to be congratulated on their work.

MATHEMATICAL FILMS.

The *First List of Films on Mathematics* recently issued by the Scientific Film Association and published in the *Gazette* of May, 1947, is a disappointment to those who are in touch with recent progress, and may be misleading to others. Differing very little from a pre-war list issued by the British Film Institute, it includes only the pioneer work, notably that of Mr. R. A. Fairthorne, in this field. About half the films should be more correctly classified under the head of Physics. The synopses are naïve to the point of inaccuracy. Those who will expect to find an exposition of the differential calculus in

Rate of Change, or the mathematics of the epicycloid and involute curves in *Transfer of Power* are doomed to disappointment.

Of more recent films mention should be made of two French films on the Polygon and the Parabola, which arrived too late to be shown at the Annual General Meeting last April ; two Czech films on the Parabola and the Ellipse, which were left with the Ministry of Education here after the Czech film visit in May ; and a group of German anti-aircraft training films, held in bond here, after rejection by the R.A.F. because they were " too mathematical ". There is also an excellent film of the mathematical section of the Palais de la Découverte in Paris, for those who cannot pay a visit.

A recent magazine film in the series " This is Britain " (No. 17) includes a section on visual aids in the teaching of mathematics. This shows students at Leicester College of Technology using some of Mr. W. W. Sawyer's models, and students of Mr. P. C. Davey at Leavesden Emergency Training College using their own visual apparatus in the primary school. Cinema-goers should look out for this film, which is likely to have " theatre release ", after which it will be available on loan from the Central Film Library. I. R. VESSELO.

An Introduction to Geometry. Based on the book of the same title by A. W. SIDDONS, M.A., and K. S. SNELL, M.A. (No. 109.) 10 pictures, with teaching notes, pp. 4. 5s. 1947. (British Instructional Films Ltd.)

The first picture shows a group of everyday objects, such as a brick, a cocoa tin and a ball ; the remaining nine show the brick in various positions, suitably marked and captioned, to illustrate the terms cuboid, vertices, edges and surfaces. The teaching notes include an excellent pre-fabricated lesson, followed by a list of questions. The whole is intended as a first lesson in geometry for age group 12.

In avoiding the usual error of overcrowding, the producers have gone to the other extreme ; the larder is bare indeed, even for these times. This experiment, for at this stage all film and film strip work is experimental, is worthy of study, if only because it gives to the lessons of experienced teachers a wider public ; but the producers must traverse the bounds of the classroom and the textbook, if they are to bring enrichment to teaching. It would seem, at least at the present stage of development, that these efforts should be devoted to such parts of the syllabus as lend themselves readily to visual treatment, rather than to attempt a complete course. I. R. V.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

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